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# Static hedging of vanilla and exotic options in a South African context

Alexis Levendis[*] and Eben Maré[†]

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**Abstract**

In this paper, we test the performance of a static hedging strategy for a long-dated European call option and European spread call option in South Africa. The stochastic volatility double jump (SVJJ) model is calibrated to historical FTSE/JSE Top40 returns to generate real-world FTSE/JSE Top40 prices at future dates. The SVJJ model is also calibrated to the FTSE/JSE (Top40) implied volatility surface in order to value the options under the risk-neutral measure. Two static hedging programs are then implemented to test their effectiveness when replicating a long-dated European call option and European spread call option. Our results indicate that static hedging is a simple, yet effective, solution when hedging non-exchange-traded options with vanilla exchange-traded options.

**Key words:** Stochastic volatility double jump model, Real-world measure, Risk-neutral measure, Calibration, Replicating portfolio, Static hedging.

## 1 Introduction

This paper is dedicated to product development and considers the sale of vanilla and exotic financial derivatives in South Africa. We focus specifically on the risk management of long-dated European call options and European spread call options for which no liquid market exists. Pricing these options is just one part of the challenge. Hedging, on the other hand, is an even bigger challenge.

Institutions wanting to sell long-dated European call options and European spread call options are faced with the challenge of buying assets to cover liabilities. In an ideal world, the portfolio manager will buy assets that match the risk sensitivities (the so-called Greeks) of the liabilities. Unfortunately, this is seldom the case as many liabilities

[*]Department of Actuarial Science, University of Pretoria, Private bag X20, Hatfield, 0028, Republic of South Africa, fax: 021 8082409, email: alexilevendis@gmail.com
[†]Department of Mathematics and Applied Mathematics, University of Pretoria
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have characteristics (longer maturities, for example) that cannot be matched perfectly by tradable assets.

[4] and [7] introduced the concept of static hedging, which replicates the value of the written option using standard exchange-traded European options with varying strikes, maturities, and fixed portfolio weights. The advantage of static hedging over traditional delta-hedging is that the hedging portfolio does not need to be rebalanced until one of the standard exchange-traded options expires.

To determine whether market makers can sell long-dated European call options and European spread call options in South Africa and manage the risks effectively, we propose a simulation-based framework to test the performance of the static hedging program under numerous market conditions. We consider the stochastic volatility double jump (SVJJ) model of [8] to simulate the underlying equity prices under the real-world probability measure, $\mathbb{P}$, which is calibrated to historical equity returns. For each simulated path under $\mathbb{P}$, we price the vanilla European call options and European spread call option under the risk-neutral measure, $\mathbb{Q}$, calibrated from traded vanilla option prices. To do this, we make use of the FFT of [11]. Finally, we test two static hedging programs based on the work of [5] and [2] to optimise the replicating portfolio weights.

The remainder of this paper is structured as follows: Section 2 introduces the SVJJ model of [8]. Section 3 focuses on the static hedging programs of [5] and [2]. Section 4 presents the static hedging results for the long-dated European call option and European spread call option, and Section 5 concludes the paper.

# 2 Stochastic volatility double jump model

This section is split into two subsections. The first subsection introduces the stochastic differential equation (SDE) for the SVJJ model that will be used to simulate real-world equity price paths. The second subsection presents the characteristic function for the SVJJ model that will be used to price the vanilla European call options.

## 2.1 SVJJ dynamics

The SVJJ model is an extension of the stochastic volatility jump (SVJ) model in [3] that adds correlated random jumps to the variance process. Under the $\mathbb{P}$-measure, the SVJJ model is given by the SDE

$$\begin{aligned} dS(t) &= (\mu - \lambda\mu_J)S(t)dt + \sqrt{v(t)}S(t)dW_S(t) + JS(t)dN(t), \\ dv(t) &= \big(\alpha - \beta v(t)\big) + \sigma_v\sqrt{v(t)}dW_v(t) + ZdN(t), \\ dW_S(t)dW_v(t) &= \rho_{S,v}dt, \end{aligned}$$

where $S(t)$ is the stock price at time $t$, $v(t)$ is the variance at time $t$, $W_S(t)$ and $W_v(t)$ are correlated Brownian motions, $N(t)$ is a Poisson process with intensity $\lambda$,

$$\mu_J = \frac{\exp\left\{\mu_S + \frac{\sigma_S^2}{2}\right\}}{1 - \rho_J\mu_V} - 1,$$

and

$$\begin{aligned} Z &\sim Exponential(\mu_V), \\ 1 + J \mid Z &\sim lognormal(\mu_S + \rho_J Z, \sigma_S^2), \end{aligned}$$

with $\mu_V$ affecting the jump size of the variance, and $\rho_J$ the correlation between the stock and variance jumps.

## 2.2 SVJJ characteristic function

The $\mathbb{Q}$-measure is obtained by calibrating a model to traded vanilla option prices. The characteristic function for the SVJJ model is an extension of the characteristic function for the models in [10] and [3]. From [12], the characteristic function for the SVJJ model, defined under the $\mathbb{Q}$-measure, is given by the product of the characteristic function in [10] and an independent jump component

$$\phi_{SVJJ}(u) = \phi_H(u)\phi_J(u),$$

where

$$\phi_H(u) = e^{iu(x(0)+rT)+C(u,T)\frac{\alpha}{\beta}+D(u,T)v(0)},$$

and

$$\begin{aligned} C(u,T) &= \beta\left[\left(\frac{Q-D_1}{2R}\right)T - \frac{2}{\sigma_v^2}\log\left(\frac{1-Ge^{-D_1T}}{1-G}\right)\right], \\ D(u,T) &= \frac{Q-D_1}{2R}\left[\frac{1-e^{-D_1T}}{1-Ge^{-D_1T}}\right], \end{aligned}$$

with

$$\begin{aligned} x(0) &= \ln S(0), \\ D_1 &= \sqrt{Q^2 - 4PR}, \\ G &= \frac{Q-D_1}{Q+D_1}, \\ P &= \frac{-u^2 - iu}{2}, \\ Q &= \beta - \rho_{x,v}\sigma_v iu, \\ R &= \frac{1}{2}\sigma_v^2. \end{aligned}$$

Furthermore,

$$\phi_J(u) = e^{-\lambda T(1+iu\mu_J)+\lambda \exp\left\{iu\mu_S+\frac{\sigma_S^2(iu)^2}{2}\right\}\nu},$$

where

$$\nu = \frac{Q + D_1}{(Q + D_1)c - 2\mu_V P} + \frac{4\mu_V P}{(D_1 c)^2 - (2\mu_V P - Qc)^2}$$
$$\times \log\left[1 - \frac{(D_1 - Q)c + 2\mu_V P}{2D_1 c}\left(1 - e^{-D1T}\right)\right],$$
$$c = 1 - iu\rho_J\mu_V.$$

The SVJJ characteristic function will be used to price vanilla European call options under the $\mathbb{Q}$-measure using the FFT of [11]. Note that the two-dimensional FFT of [11] for European spread call options reduces to the one-dimensional case for vanilla European call options when the second asset price is set to zero.

In the next section, we introduce the static hedging programs of [5] and [2].

# 3 Static hedging

This section is split into two subsections and introduces two static hedging programs that can be used to optimise the instrument weights in the replicating portfolio. The first subsection introduces the static hedging program of [5], and the second subsection focuses on the static hedging program of [2].

## 3.1 Choie and Novometsky optimisation

The static hedging program of [5] seeks to minimise the cost of setting up the replicating portfolio, subject to the value of the replicating portfolio being greater than or equal to the value of the target option at some future date. Mathematically, this can be expressed as

$$\min_{\boldsymbol{B}} \sum_{i=1}^{n} C(i)B(i),$$

subject to

$$\sum_{i=1}^{n} F(i,j)B(i) \geq Y(j), \quad j = 1, 2, ..., m,$$

where

$C(i)$ is the current price of the $i^{th}$ instrument;

$B(i)$ is the number of units of the $i^{th}$ instrument;

$F(i,j)$ is the future price of the $i^{th}$ instrument in state $j$; and

$Y(j)$ is the future price of the target option in state $j$.

In the next subsection, we introduce the static hedging program of [2].

### 3.2 Armstrong et al. optimisation

The static hedging program of [2] seeks to minimise the difference between the value of the replicating portfolio and the target option at some future date, subject to the cost of the replicating portfolio being less than or equal to the initial wealth, i.e., the premium received from the written option. Mathematically, this can be written as

$$\min_{\boldsymbol{B}} \sum_{j=1}^{m} \Big( Y(j) - \sum_{i=1}^{n} F(i,j)B(i) \Big)^2,$$

subject to

$$\sum_{i=1}^{n} C(i)B(i) \leq w,$$

where

$C(i)$ is the current price of the $i^{th}$ instrument;

$B(i)$ is the number of units of the $i^{th}$ instrument;

$F(i,j)$ is the future price of the $i^{th}$ instrument in state $j$;

$Y(j)$ is the future price of the target option in state $j$; and

$w :=$ the initial wealth.

In the next section, we present the static hedging results for a 5-year European call option and 1-year European spread call option based on the static hedging programs of [5] and [2].

## 4 Results

This section is split into four subsections. The first subsection presents the calibration results for the SVJJ model to the FTSE/JSE Top40 index under the $\mathbb{P}$-measure. The second subsection contains the calibration results for the SVJJ model to the FTSE/JSE Top40 implied volatility surface under the $\mathbb{Q}$-measure. The third subsection presents the static hedging results for a 5-year vanilla European call option written on the FTSE/JSE Top40 index; and, lastly, the fourth subsection shows the static hedging results for an arbitrary 1-year European spread call option.

### 4.1 SVJJ $\mathbb{P}$-measure calibration

The first step in setting up the simulation-based framework for static hedging is to calibrate the SVJJ model under the $\mathbb{P}$-measure to forecast future prices for the FTSE/JSE Top40.

Figure 1 below shows the historical closing prices for the FTSE/JSE Top40 index from 30 June 1995 to 30 October 2020.

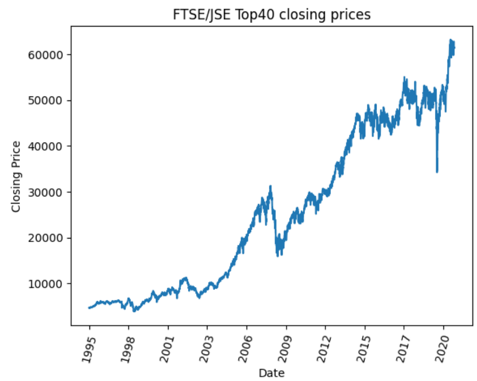


Figure 1: FTSE/JSE Top40 closing prices

Using the efficient method of moments (EMM) of [9], we calibrated the SVJJ model to daily returns from the FTSE/JSE Top40 index over the period 30 June 1995 to 30 October 2020. The calibrated parameters and goodness-of-fit statistic are shown in Table 1 below.

| **Parameter** | **FTSE/JSE Top40** |
|---|---|
| $\mu$ | 0.1180 |
| $\alpha$ | 0.2888 |
| $\beta$ | 6.0176 |
| $\sigma_v$ | 0.4543 |
| $\rho_{x,v}$ | -0.9374 |
| $\lambda$ | 4.7284 |
| $\sigma_S$ | 0.0137 |
| $\mu_V$ | 0.0077 |
| $\rho_J$ | -0.3052 |
| Critical value | 5.1022 |
| $\chi^2_{0.05}$ | 5.9915 |

Table 1: SVJJ $\mathbb{P}$-parameters for FTSE/JSE Top40

In the calibration, we set $\mu_S = 0$, since this parameter is generally insignificant and poorly identified as explained by [1].

From Table 1, the SVJJ model expects between four and five jumps per year on average. Furthermore, note the strong negative relationship between the stock and variance processes, and also the negative correlation between the stock and variance jumps. As a result, the model will produce a negative skew for the FTSE/JSE Top40.

Table 2 below compares the first four corresponding empirical values observed in the moments from the SVJJ model with the daily returns from the FTSE/JSE Top40 index.

| Statistic | FTSE/JSE Top40 index | SVJJ model |
|---|---|---|
| Mean | 0.0385% | 0.0406% |
| Std dev | 1.3290% | 1.1410% |
| Skewness | -0.4369 | -0.2418 |
| Kurtosis | 9.4344 | 5.0463 |

Table 2: SVJJ model daily statistics for the FTSE/JSE Top40

The results indicate that the SVJJ model captures the mean and standard deviation well for the FTSE/JSE Top40 index, but underestimates the skewness and kurtosis. However, the goodness-of-fit statistic in Table 1 suggests that the SVJJ model is not rejected at a 5% level of significance. The SVJJ model is, therefore, a plausible data-generating model for the FTSE/JSE Top40 index.

Figure 2 below compares the desnity under the fitted SVJJ model with a kernel density estimate of the observed FTSE/JSE Top40 returns.

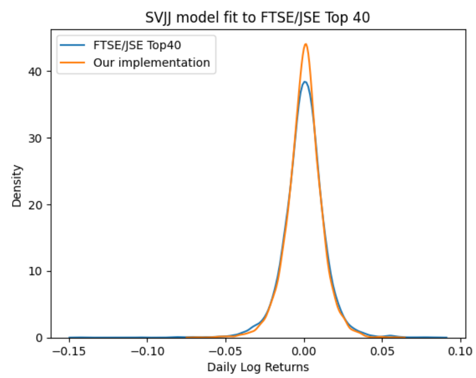


Figure 2: SVJJ model versus FTSE/JSE Top40 desnities

The SVJJ model fits the historical distribution well. The $\mathbb{P}$-SVJJ model will be used to generate real-world sample paths for the FTSE/JSE Top40 index using Monte Carlo simulation. For each real-world path, the value of the written vanilla European call option, and values of the replicating options, must be calculated under the $\mathbb{Q}$-measure. This is the focus of the next subsection.

## 4.2 SVJJ $\mathbb{Q}$-measure calibration

For the purpose of this paper, we assume a constant risk-free interest rate, $r = 7\%$. Using the SVJJ characteristic function in Section 2.2 and the FFT of [11] for European spread call options (reduced to one dimension by setting the second asset price to zero), The SVJJ model was calibrated to the FTSE/JSE Top40 implied volatility surface on 16 November 2020. Table 3 below shows the calibrated parameters. Note that a tilde has been placed over each parameter to distinguish the fitted $\mathbb{Q}$-parameters from the estimated $\mathbb{P}$-parameters.

| Parameter | FTSE/JSE Top40 |
|---|---|
| $r$ | 0.0700 |
| $\tilde{\alpha}$ | 0.0333 |
| $\tilde{\beta}$ | 0.9995 |
| $\tilde{\sigma_v}$ | 0.3827 |
| $\rho_{\tilde{x,v}}$ | -0.9205 |
| $\tilde{\lambda}$ | 0.0583 |
| $\tilde{\sigma_S}$ | 0.0058 |
| $\tilde{\mu_V}$ | 0.0058 |
| $\tilde{\rho_J}$ | 0.0097 |

Table 3: Fitted SVJJ $\mathbb{Q}$-parameters for FTSE/JSE Top40

Note that the $\mathbb{Q}$-parameters in Table 3 differ from the $\mathbb{P}$-parameters in Table 1. [13] explain that the returns distribution resulting from calibration to option prices can differ significantly from the historical returns distribution. The authors mention that a possible solution is to combine option prices and historical returns in the calibration procedure in order to minimise the discrepancy between the real-world and risk-neutral distributions. However, this generally leads to larger errors between the model prices and option prices.

Figure 3 below shows the fit of the SVJJ model to the FTSE/JSE Top40 implied volatility surface on 16 November 2020.

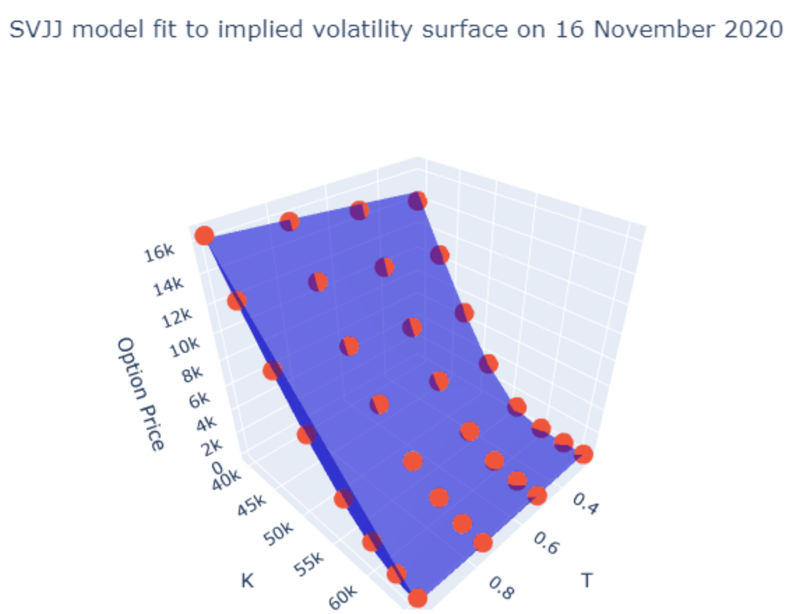


Figure 3: SVJJ fit to FTSE/JSE Top40 implied volatility surface

The red dots represent the market quotes for FTSE/JSE Top40 European call options on 16 November 2020, and the blue surface represents the SVJJ model prices. Note that the SVJJ model reprices the exchange-traded FTSE/JSE Top40 options well.

Before tackling the static hedging experiment, we first compared our implementation of the FFT for arbitrary European call options with the results obtained from a Monte Carlo simulation with 100,000 samples. Efficient pricing is important for the static hedging experiment, since option values must be calculated for multiple real-world paths. Monte Carlo simulation is computationally too expensive. Table 4 below compares the European call option prices for each numerical method.

Table 4: Monte Carlo (MC) and FFT European call option prices under SVJJ model with $S(0) = 100,\ r = 0.1,\ v(0) = 0.04,\ \beta = 1,\ \alpha = 0.04,\ \sigma_v = 0.05,\ \rho_{x,v} = -0.5,\ \lambda = 5,\ \mu_S = 0,\ \sigma_S = 0.01,\ \rho_J = -0.3,\ \mu_V = 0.02,\ N = 256,\ \bar{u} = 40,\ \epsilon_1 = -3,\ \epsilon_2 = 1,\ T = 1.$

| $K$ | MC Price | FFT Price | Absolute Difference |
|---|---|---|---|
| 20 | 81.906164 | 81.903234 | 0.002930 |
| 30 | 72.872779 | 72.855205 | 0.017574 |
| 40 | 63.773907 | 63.811614 | 0.037707 |
| 50 | 54.788865 | 54.795772 | 0.006907 |
| 60 | 45.908156 | 45.885217 | 0.022939 |
| 70 | 37.259369 | 37.253208 | 0.006161 |
| 80 | 29.160481 | 29.176272 | 0.015791 |
| 90 | 21.995527 | 21.975674 | 0.019853 |

Table 4 confirms that our implementation of the FFT was accurate. The FFT prices a single option in approximately 2.61 seconds, compared to 215.50 seconds in the case of Monte Carlo simulation.

For the static hedging experiment, we first consider the sale of a 5-year at-the-money vanilla European call option on the FTSE/JSE Top40 index on 16 November 2020. To hedge the sold option, we set up a static hedging portfolio consisting of 3-month, 6-month, 9-month, and 12-month exchange-traded FTSE/JSE Top40 index options and cash. Note that the FTSE/JSE Top40 index option prices on 16 November 2020 are readily available from the option price surface in Figure 3. Their future prices can also be obtained by simulating real-world variations for the state variables from the $\mathbb{P}$-SVJJ model, and substituting these values in the $\mathbb{Q}$-SVJJ model.

The next subsection presents the static hedging results for the vanilla European call option.

## 4.3 Static hedging performance for vanilla call option

The results in this subsection show the static hedging performance for a written 5-year at-the-money vanilla European call option. Since the longest maturity for the exchange-traded FTSE/JSE Top40 options is generally 1 year, the replicating portfolio will need to be rolled as the options expire. As explained by [5], once the shortest dated option (3 months in our case) expires, the proceeds from the sale of the replicating portfolio will be used to purchase a new portfolio consisting of cash and 3-month, 6-month, 9-month, and 12-month FTSE/JSE Top40 options. This process repeats until the expiry of the written option; in this case, the 5-year FTSE/JSE Top40 European call option. It is important to note that each hedging interval is only for a period of 3 months.

Calculating the distribution of values for the 3-month option at the 3-month mark is simply $\max\big(S(0.25) - K, 0\big)$, where $S(0.25)$ are the real-world forecasts for the FTSE/JSE Top40 index from the $\mathbb{P}$-SVJJ model 3 months ahead, and $K$ is the strike price of the option. The valuation of the 6-month, 9-month, and 12-month options at the 3-month mark is more complicated.

At the 3-month mark, the 6-month, 9-month, and 12-month options that were bought at inception have maturities of 3 months, 6 months, and 9 months respectively. Their values can be calculated by substituting the real-world forecasts, $S(0.25)$ and $v(0.25)$, into the SVJJ characteristic function along with the $\mathbb{Q}$-SVJJ parameters in Table 3. The FFT of [11] can then be used to calculate the option values.

The valuation of the 5-year FTSE/JSE Top40 index option at the 3-month mark follows a similar process, where the maturity of the option at this point is 4.75 years. This process gets repeated every quarter.

Table 5 below details the information for the written European call option on 16 November 2020.

Table 5: Market information for European call option on 16 November 2020

| | |
|---|---|
| Option sale date | 16 November 2020 |
| Underlying | FTSE/JSE Top40 index |
| $S(0)$ | 52552 |
| $K$ | 52552 |
| $T$ | 5 |

Figure 4 below shows the estimated real-world density for the FTSE/JSE Top40 index at $t = 0.25$ generated from the $\mathbb{P}$-SVJJ model in Table 1 with 10,000 Monte Carlo samples.

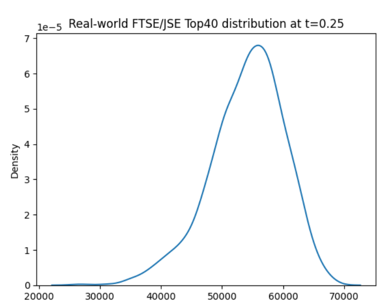


Figure 4: Real-world FTSE/JSE Top40 distribution at $t = 0.25$

The option writer has the entire FTSE/JSE Top40 option price surface at his disposal when faced with the challenge of hedging the 5-year at-the-money European call option. The seller's aim is to find the optimal quantity for each exchange-traded option on the option price surface to hedge his position at $t = 0.25$.

The static hedging results based on the optimisation routines of [5] and [2] are shown below.

**Choie and Novometsky optimisation**

Using the optimisation program of [5] discussed in Section 3.1, Figure 5 below shows the optimised quantities on 16 November 2020 for the exchange-traded options based on the real-world distribution for the FTSE/JSE Top40 index at $t = 0.25$ in Figure 4.

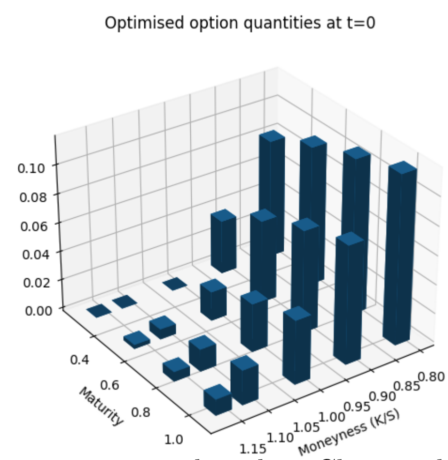


Figure 5: Replicating option quantities based on Choie and Novometsky optimisation

Note that the option quantities were calculated for the first hedging interval, $t = 0$ to $t = 0.25$, and expressed as a percentage. The replicating portfolio is skewed more towards in-the-money options, and the optimised cash balance was $R6,500$. The cost of setting up the replicating portfolio on 16 November 2020 was $R14,170$. On the other hand, the premium received from the sale of the 5-year at-the-money option was $R13,656$, which was calculated from the $\mathbb{Q}$-SVJJ model, i.e., consistent with the market prices on 16 November 2020. Therefore, the option writer recorded an upfront loss of $R13,656 - R14,170 = -R514$.

Figure 6 below compares the value of the replicating portfolio based on the option quantities in Figure 5 and a cash balance of $R6,500$ with the value of the target option at $t = 0.25$.

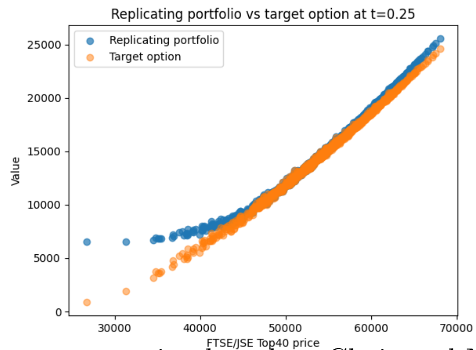


Figure 6: Portfolio versus target option based on Choie and Novometsky optimisation

Next, we show the static hedging results based on the optimisation routine of [2]. The results will be discussed thereafter.

## Armstrong et al. optimisation

Using the optimisation routine of [2] discussed in Section 3.2, Figure 7 below shows the optimised quantities for the exchange-traded options based on the estimated real-world density for the FTSE/JSE Top40 index at $t = 0.25$ in Figure 4.

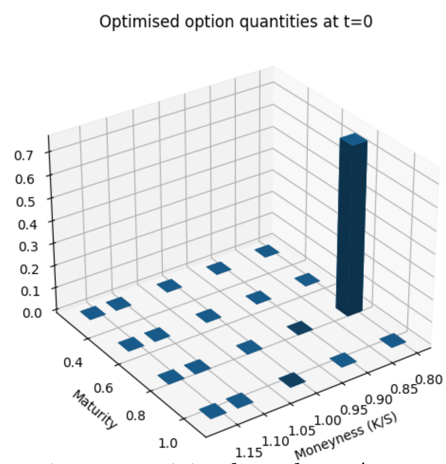


Figure 7: Replicating option quantities based on Armstrong et al. optimisation

Once again, the option quantities were calculated for the first hedging interval, $t = 0$ to $t = 0.25$, and expressed as a percentage. Note that the optimisation returns a single option on the FTSE/JSE Top40 option price surface, which is significantly different from the results obtained by using the routine in [5].

The cost of setting up the replicating portfolio on 16 November 2020 was $R13,656$, which is exactly equal to the premium received. The optimised cash balance was $R6,425$.

Figure 8 below compares the value of the replicating portfolio based on the option quantities in Figure 7 and a cash balance of $R6,425$ with the value of the target option at $t = 0.25$.

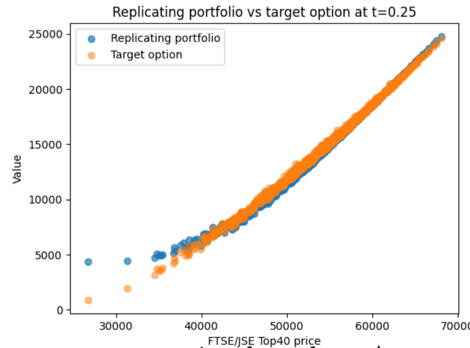


Figure 8: Portfolio versus target option based on Armstrong et al. optimisation

The results are discussed next.

## Results discussion

Recall that the static hedging program of [5] seeks to minimise the cost of setting up the replicating portfolio, subject to the value of the replicating portfolio being greater than or equal to the value of the target option at some future date, i.e, 3 months in our case. Figure 6 illustrates that the constraint was met – the value of the replicating portfolio was greater than or equal to the value of the target option in each of the real-world FTSE/JSE Top40 states at $t = 0.25$.

Figure 5 showed that the optimisation in [5] produced replicating option quantities across most of the FTSE/JSE Top40 option price surface, with the quantities skewed more towards in-the-money options. A possible explanation for this is that the value of the target option is quite sensitive to price movements in the FTSE/JSE Top40, i.e., delta. Hence, the replicating portfolio is skewed more towards in-the-money options since they have the highest delta.

The cost of setting up the replicating portfolio was slightly more expensive than the upfront premium received; $R14,170$ versus $R13,656$. The option writer, therefore, recorded an upfront loss.

The static hedging performance based on the optimisation of [5] also deteriorated at the tails of the FTSE/JSE Top40 distribution.

The second routine tested was the optimisation of [2]. Recall that this optimisation seeks to minimise the difference between the value of the replicating portfolio and the target option at some future date (3 months in our case), subject to the cost of the replicating portfolio being less than or equal to the premium received from the written option. The optimisation returned a cost that matched the premium from the written option exactly.

Figure 7 showed that the optimisation of [2] returned a single in-the-money option on the FTSE/JSE Top40 option price surface. Based on this option and a cash balance of $R6,425$, Figure 8 showed that there were instances where the value of the replicating portfolio was less than the value of the target option at $t = 0.25$.

In summary, the choice of optimisation routine can produce significantly different quantities for the instruments in the replicating portfolio. The replicating portfolio based on the optimisation of [5] might be slightly more expensive to set up than the premium received, but ensures that the value of the replicating portfolio is greater than or equal to the value of the target option for the state variables considered at some future date. Alternatively, the cost of setting up the replicating portfolio based on the optimisation of [2] is equal to the upfront premium received. The risk is that the value of the replicating portfolio might be less than the value of the target option at some future date.

Considering the complexity of hedging an option written on the FTSE/JSE Top40, which exhibits factors such as stochastic volatility and jumps, the static hedging approach is simple and shows promising results.

In the next subsection, we test the static hedging performance for an arbitrary 1-year European spread call option.

### 4.4 Static hedging performance for spread call option

This subsection presents the static hedging results for an arbitrary 1-year European spread call option, hedged with vanilla FTSE/JSE Top40 European call options. Note that the underlying instrument used to hedge the spread option is not necessarily the same as the underlying instruments in the spread option.

Figure 9 below shows a kernel density estimate of the real-world distribution for the spread ($S_1$-$S_2$) at $t = 0.25$, generated from the $\mathbb{P}$-SVJJ model in Table 1, with $S_1(0) = 100$, $S_2(0) = 96$, $\mu_1 = 0.13$, and $\mu_2 = 0.11$. Note that we have just scaled the FTSE/JSE Top40 price by assigning different starting values and expected returns to $S_1$ and $S_2$, while keeping the remaining parameters unchanged.

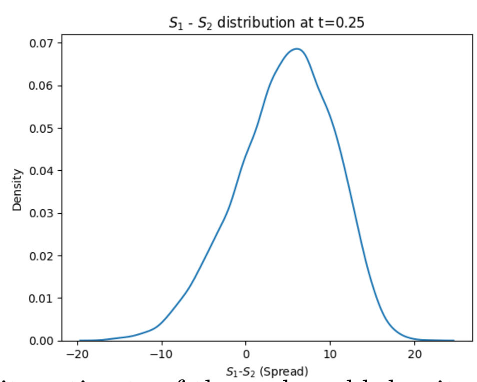


Figure 9: Kernel density estimate of the real-world density for $S_1$ - $S_2$ at $t = 0.25$

We further set the correlation between the Brownian motions driving $S_1$ and the FTSE/JSE Top40 equal to 1, and similar for $S_2$ and the FTSE/JSE Top40.

Figure 10 below shows the relationship between the FTSE/JSE Top40 price and the spread ($S_1$-$S_2$) based on 10,000 simulations at $t = 0.25$.

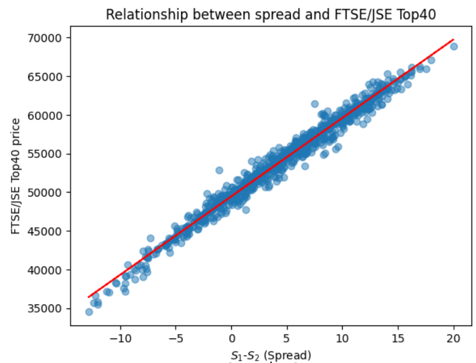


Figure 10: Relationship between FTSE/JSE Top40 and $S_1 - S_2$ at $t = 0.25$

It is important to note that a linear relationship between the price of the underlying instrument used to hedge the spread option and the spread ($S_1$-$S_2$) must exist in order for the static hedge to work. Correlation is a substantial risk when hedging European spread call options with vanilla European call options.

Table 6 details the market information for the spread call option on 16 November 2020.

Table 6: Market information for European spread call option on 16 November 2020

| | |
|---|---|
| Option sale date | 16 November 2020 |
| Underlying hedge instrument | FTSE/JSE Top40 index |
| $S_{Top40}(0)$ | 52552 |
| $S_1(0)$ | 100 |
| $S_2(0)$ | 96 |
| $K$ | 3 |
| $T$ | 1 |

The static hedging results for the European spread call option are discussed next.

## Choie and Novometsky optimisation

Using the optimisation routine of [5], Figure 11 below shows the optimised quantities for the FTSE/JSE Top40 options based on the density in Figure 9.

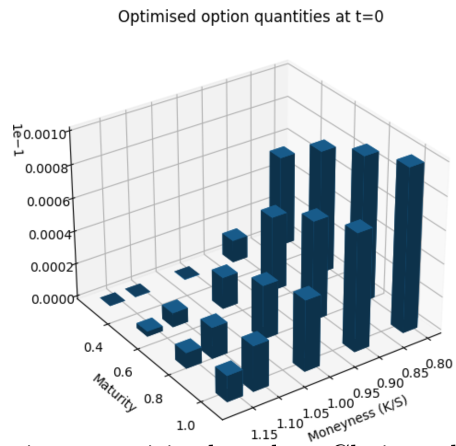


Figure 11: Replicating option quantities based on Choie and Novometsky optimisation

The option quantities were calculated for the first hedging interval, $t = 0$ to $t = 0.25$. Note that the replicating portfolio is skewed more towards in-the-money options, and the optimised cash balance was $R4$. The European spread call option value was calculated from the three-factor stochastic volatility model of [6], and returned an option premium of $R7.48$. The cost of setting up the replicating portfolio based on the optimisation of [5] was $R10.23$. Hence, the option writer recorded an upfront loss of $R7.48 - R10.23 = -R2.75$.

Figure 12 below compares the value of the replicating portfolio with the value of the European spread call option at $t = 0.25$.

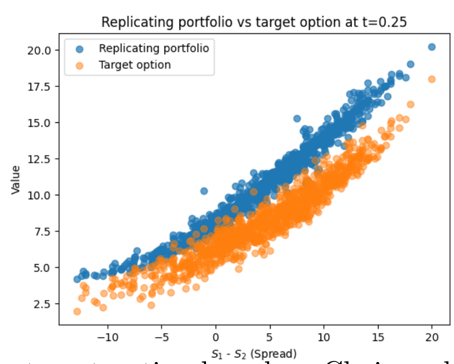


Figure 12: Portfolio versus target option based on Choie and Novometsky optimisation

Note that the value of the replicating portfolio is greater than or equal to the value of the European spread call option for all states considered at $t = 0.25$.

The static hedging results for the European spread call option based on the optimisation of [2] are discussed next.

## Armstrong et al. optimisation

Using the optimisation program of [2], Figure 13 below shows the optimised quantities for FTSE/JSE Top40 options based on the real-world spread density in Figure 9.

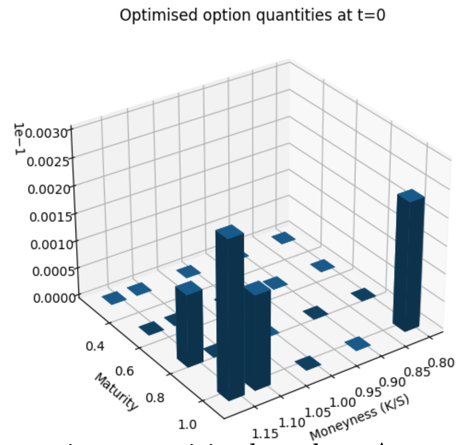


Figure 13: Replicating option quantities based on Armstrong et al. optimisation

Again, the option quantities were calculated for the first hedging interval, $t = 0$ to $t = 0.25$. Note that the optimisation of [2] returns only a small number of options on the FTSE/JSE Top40 surface. The optimised cash balance was $R3$, and the cost of setting up the replicating portfolio was $R7.48$, i.e., equal to the premium received from the written European spread call option.

Figure 14 compares the value of the replicating portfolio with the value of the European spread call option at $t = 0.25$.

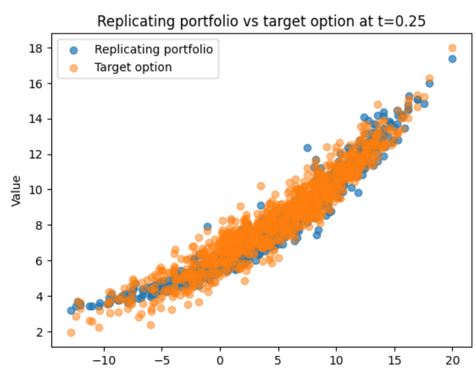


Figure 14: Portfolio versus target option based on Armstrong et al. optimisation

Note that there are instances where the value of the replicating portfolio is less than the value of the European spread call option at $t = 0.25$.

The results are discussed next.

**Results discussion**

For the European spread call option, the optimisation of [5] returned a replicating portfolio with option quantities spanning almost the entire FTSE/JSE Top40 option price surface. Conversely, the optimisation of [2] returned only a small number of options on the FTSE/JSE Top40 option price surface.

Based on the optimisation of [5], the cost of setting up the replicating portfolio was $R10.23$. The premium received from the written option was $R7.48$, hence, the option writer recorded an upfront loss.

The cost of the replicating portfolio based on the optimisation of [2] was exactly equal to the premium received. However, the optimisation did not guarantee a replicating portfolio value that was greater than or equal to the value of the European spread call option at a future date.

It is important to note that the static hedge will only work if the price of the hedging instrument is strongly and positively correlated with the spread generated by the two underlying instruments in the spread option. For European spread call options written on two stocks that form part of the FTSE/JSE Top40 index, this may very well be the case. We suggest this as an area for future research.

## 5 Conclusion

The purpose of this paper was to link the $\mathbb{P}$- and $\mathbb{Q}$-probability measures for stochastic volatility models. To achieve this, we considered a portfolio risk management problem, i.e., static hedging of a long-dated European call option and European spread call option in South Africa.

In risk management applications, the $\mathbb{P}$-measure is typically used to generate real-world events that can affect the value of a portfolio. Therefore, we calibrated the $\mathbb{P}$-SVJJ model to historical FTSE/JSE Top40 returns in order to generate share price and volatility shocks.

Pricing derivatives is done under the $\mathbb{Q}$-measure due to the principle of no arbitrage. Therefore, we calibrated the $\mathbb{Q}$-SVJJ model to the FTSE/JSE Top40 option price surface in order to price the options.

The link between $\mathbb{P}$ and $\mathbb{Q}$ was introduced when we simulated future option values. The process followed was to consider the current FTSE/JSE Top40 option price surface on 16 November 2020 and to apply shocks to the underlying state variables (stock price and volatility), where the shocks were produced by the $\mathbb{P}$-SVJJ model over a 3-month period. We then revalued the options for each of the scenarios using the $\mathbb{Q}$-SVJJ model to produce a distribution of option values at $t = 0.25$. Note that this process is similar to VaR.

For the static hedge of the long-dated European call option, we applied the optimisation programs of [5] and [2] to calculate the instrument weights in the replicating portfolio. The optimisation of [5] produced weights across most of the FTSE/JSE option price surface and guaranteed that the value of the replicating portfolio was greater than or equal to the value of the long-dated European call option at a future date. However, the cost of setting up the replicating portfolio was more expensive than the premium received, leading to an upfront loss. The optimisation of [2] returned a single option on the FTSE/JSE option price surface with cost equal to the premium received, but did not guarantee a replicating portfolio value that was greater than or equal to the value of the long-dated European call option at a future date. The difference between the two optimisation routines is the timing of the loss.

The static hedging results for the European spread call option were similar to the results for the long-dated European call option. It is important to note that the key risk when hedging a European spread call option with vanilla European call options is correlation. An area for future research might be to consider a European spread call option that is struck on two stocks that form part of the FTSE/JSE Top40 index.

Static hedging shows promising results in the South African market and option writers may find that static hedging provides a cheaper and more effective solution than traditional delta-hedging.

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