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Estimating the dependence parameter in bivariate extreme value statistics through a Bayesian approach

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Abstract

In bivariate extreme value theory, the estimation of the dependence parameter plays an important role. In this study we consider the bivariate tail model given by Ledford and Tawn [10] to estimate the dependence parameter. The difference however is that a Bayesian (instead of a classical) approach is considered when estimating the dependence parameter. The dependence parameter η lies between 0 and 1. For $\eta < 1$ the variables are asymptotically independent whereas $\eta = 1$ indicates asymptotic dependence. In this study four prior distributions, restricted between 0 and 1, are considered. Thus, the posteriors also enforce strict boundaries on the dependence parameter, ensuring that it does not go beyond 1. This restriction has been neglected in existing classical literature. Further contributions are made regarding mathematical formulas to calculate some properties and probabilities of the dependence parameter.

Key words: Bayesian estimation, beta prior, dependence parameter, bivariate extremes.

1 Introduction

Bivariate extreme value theory relies heavily on the estimation of the dependence parameter. The most well-known and popular methods for estimating the dependence parameter relies on the concept of asymptotic dependence in the joint tail. Ledford and Tawn [10] explain that these methods are restrictive and might be problematic in the case of asymptotic independence. They introduced a bivariate tail model with dependence index $\eta \in (0, 1]$ which they refer to as the coefficient of tail dependence. This model is discussed in more detail in Section 2. The model was further explored by [6] and [4].

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In this paper we consider the approach introduced by [10] by taking a Bayesian (rather than a classical) approach. The advantage of the Bayesian approach is that prior information of η is built into the estimation process. Our aim is to show that a Bayesian approach, with priors, on η, that are restricted between 0 and 1, is suitable to estimate the dependence parameter η. Our approach will be compared to some well-known classical estimators.

The paper is structured as follows: Section 2 presents the joint tail model and some well-known classical tail parameter estimators. Section 3 introduces a Bayesian approach with four priors and posteriors. The Bayesian approach is assessed through a simulation study in Section 4 and real-life examples in Section 6. Section 5 is dedicated to theoretical derivations to express our estimate of η through mathematical formulas which simplify the estimation process in real life situations. Finally two appendices accompany the paper; Appendix A is presented at the end of the paper while Appendix B is published in the supplementary material of the paper.

2 The joint tail model of Ledford and Tawn [10]

Consider the joint tail model $P(X > r, Y > r) \sim \mathscr{L}(r)r^{1/\eta}$ as $r \to \infty$ for a random vector (X, Y), where $\mathscr{L}$ is a slowly varying function with $\eta \in (0, 1]$. The random vector (X, Y) has unit Fréchet or Pareto marginals, also see [4]. For $\eta < 1$ the random variables are asymptotically independent (η then measures the amount of tail dependence within the asymptotical independence) and for $\eta = 1$ the random variables are asymptotically dependent (see [10]). Suppose that $S = \min(X, Y)$ then

$$P(S > r) = P(X > r, Y > r) \sim \mathscr{L}(r)r^{1/n}, r \to \infty. \tag{1}$$

Equation (1) is the survival function of the Pareto type distribution with extreme value index (EVI) $\eta > 0$. It can be shown that the Pareto type distribution follows a peak over threshold model with the following limit, as $n \to \infty$:

$$P(S/u > r|S > u) \to \bar{H}_\eta(r) = r^{-1/\eta}, r > 1, \tag{2}$$

where u is a suitable threshold (see, for example, [3] and [5]). It is well known that the threshold, u, in (2) should be chosen as high as possible to control the bias of η. On the other hand, if the threshold is chosen too high the estimation variance becomes out of control. This is a complicated trade off and attention should be given to the decision as to where the threshold is chosen. This paper will not discuss methods for choosing a threshold.

The marginal distributions are first standardized to unit Fréchet or unit Pareto distributions, see [6]. In this study we decided to transform the marginals to unit Pareto distributions. Suppose that $R_{X,i}$ is the rank of $X_i, i = 1, \ldots, n$ and $R_{Y,i}$ is the rank of $Y_i, i = 1, \ldots, n$, the following transformation method of [4] is followed:

$$S_i = \min\left(-\frac{1}{\log\left(\frac{R_{X,i}}{n+1}\right)}, -\frac{1}{\log\left(\frac{R_{Y,i}}{n+1}\right)}\right).$$

η can be estimated through any well know extreme value index (EVI) estimator. One of the most famous and well-known estimators of the EVI is the Hill estimate (see [9]):

$$\hat{\eta}_{Hill} = \frac{1}{k}\sum_{j=1}^{k} \log S_{n-j+1,n} - \log S_{n-k,n}, \tag{3}$$

where k represents the number of observations above the threshold u. The Hill estimator has proven to be problematic in the sense that it has severe bias. Numerous papers have been devoted to the improvement of the Hill estimate, for example the generalised Hill estimator given (see [2]):

$$\hat{\eta}_{Gen-Hill} = \frac{1}{k}\sum_{j=1}^{k} \log UH_{j,n} - \log UH_{k+1,n}, \tag{4}$$

where $UH_{j,n} = S_{j-n,n}H_{j,n}$ and $H_{j,n}$ is the Hill estimator from (3).

Another improvement is the biased reduced Hill estimator ($\hat{\eta}_{BRHill}$) which computes biased reduced estimates from a quantile view. This method is discussed in detail in [2], Section 4.2.1.

3 Estimating the dependence parameter through a Bayesian approach

In this section we consider estimating the dependence parameter, η, through a Bayesian approach. The benefit of using a Bayesian approach is that prior knowledge can be built into the estimation process. In this case we consider prior distributions on η that are restricted to $(0, 1]$. The prior distributions that we will consider are the $Beta(a, b)$, $Uniform[0, 1]$, $Topp\ Leone(\alpha)$ and truncated $Exponential(\lambda, b)$ distributions. Two well-known reference priors (the maximum data information prior (MDI) and Jeffreys' prior) were also considered but are exempted from this discussion because they do not comply with the $(0, 1]$ restriction on η. The four priors are given below:

1. $Beta(a, b)$ prior: $\pi(\eta|a, b) \propto \eta^{a-1}(1-\eta)^{b-1}, a, b > 0, 0 < \eta < 1$. Since no information is known about a and b, vague exponential priors are assumed on the hyperparameters a and b, $\pi(a) \sim Exp(0.001)$, $\pi(b) \sim Exp(0.001)$. Thus, the joint prior is:

$$\pi_{Beta}(\eta, a, b) \propto \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \mathrm{e}^{-0.001(a+b)}\eta^{a-1}(1-\eta)^{b-1}.$$

2. $Uniform[0, 1]$ prior:

$$\pi_{Uni}(\eta) \propto 1, 0 < \eta < 1.$$

3. $Topp\ Leone(\alpha)$ prior: $\pi(\eta|\alpha) \propto \alpha\eta^{\alpha-1}(1-\eta)(2-\eta)^{\alpha-1}, \alpha > 0, 0 < \eta < 1$. A vague exponential prior is assumed on α, $\pi(\alpha) \sim Exp(0.001)$. The joint prior is:

$$\pi_{TL}(\eta, \alpha) \propto \alpha\eta^{\alpha-1}(1-\eta)(2-\eta)^{\alpha-1}\mathrm{e}^{-0.001\alpha}.$$

   See [1] for more information on the Topp Leone distribution.

4. *Truncated exponential*$(\lambda, b=1)$ prior: $\pi(\eta,\lambda) \propto \lambda e^{-\lambda\eta}[1-e^{-\lambda}]^{-1}, 0<\eta<1, \lambda>0$. Again, a vague exponential prior is assumed on λ, $\pi(\lambda) \sim Exp(0.001)$. The joint prior is:

$$\pi_{TEXP}(\eta,\lambda) \propto \lambda e^{-\lambda\eta}[1-e^{-\lambda}]^{-1}e^{-0.001\alpha}.$$

The posterior distribution is constructed by multiplying the prior distribution by the likelihood, $\pi(\eta|\mathbf{t}) \propto \pi(\eta)L(\eta|t_i)$ where $L(\eta|t_i)$ is the likelihood of the Pareto distribution and $t_i = s_i/u, i = 1,\ldots,N$, representing the N relative excesses above the threshold (u). The four posterior distributions are derived below by using the above priors respectively:

1.

$$\begin{aligned}\pi_{Beta}(\eta,a,b|\mathbf{t}) &\propto \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}e^{-0.001(a+b)}\eta^{a-1}(1-\eta)^{b-1}\prod_{i=1}^{N}\frac{1}{\eta}t_i^{-1-1/\eta}\\ &\propto \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}e^{-0.001(a+b)}\eta^{a-1-N}(1-\eta)^{b-1}e^{-\frac{1}{\eta}\sum_{i=1}^{N}\log(t_i)}, \quad (5)\end{aligned}$$

with $t_i > 1, 0<\eta<1, a, b>0$.

2.

$$\pi_{Uni}(\eta|\mathbf{t}) \propto \prod_{i=1}^{N}\frac{1}{\eta}t_i^{-1-\frac{1}{\eta}} \propto \frac{1}{\eta^N}e^{-\frac{1}{\eta}\sum_{i=1}^{N}\log(t_i)}, t_i>1, 0<\eta<1. \quad (6)$$

3.

$$\pi_{TL}(\eta,\alpha|\mathbf{t}) \propto \alpha\eta^{\alpha-1}(1-\eta)(2-\eta)^{\alpha-1}e^{-0.001\alpha}\frac{1}{\eta^N}e^{-\frac{1}{\eta}\sum_{i=1}^{N}\log(t_i)}, \quad (7)$$

for $t_i > 1, 0<\eta<1, \alpha>0$.

4.

$$\pi_{TEXP}(\eta,\lambda|\mathbf{t}) \propto \lambda e^{-\lambda\eta}[1-e^{-\lambda}]^{-1}e^{-0.001\lambda}\frac{1}{\eta^N}e^{-1/\eta\sum_{i=1}^{N}\log(t_i)}, \quad (8)$$

for $t_i > 1, 0<\eta<1, \lambda>0$.

In this study the estimates of the unknown parameter, η, are calculated under the squared error loss functions. Thus, the mean of the posteriors are estimates of the unknown parameter, η. Given as $\hat{\eta}_{Beta}$, $\hat{\eta}_{Uni}$, $\hat{\eta}_{TL}$ and $\hat{\eta}_{TEXP}$ respectively.

In the next section a small simulation study is conducted to investigate the performance, of the four Bayes estimates of η, in terms of bias and RMSE. Our estimates are also compared to the classical Hill, generalised Hill and the biased reduced Hill estimates from Section 2. The classical estimates were computed using the ReIns package in R, https://cran.r-project.org/web/packages/ReIns/ReIns.pdf.

4 Simulation study

In this section we follow the simulation study of [4]. We simulate 100 samples of sizes $n = 200$ and $n = 500$ from the bivariate normal and the bivariate Morgenstern with various parameter values. For each simulation the dependence parameter η is estimated from the posterior distributions in Section 3. For each of the 100 samples, 2000 η values are simulated from Equations (5) - (8) respectively. The mean of the 2000 simulated η's is taken as an estimate of η, given respectively as $\hat{\eta}_{Beta}$, $\hat{\eta}_{Uni}$, $\hat{\eta}_{TL}$ and $\hat{\eta}_{TEXP}$.

In this simulation study we consider the estimates of η over four different threshold values; the 10th, 20th, 50th and 70th percentiles respectively (for each of the 100 samples). Thus, considering the top 90% of the ordered observations (almost the whole dataset) up to only the top 30% of the ordered observations. The results are summarised over the different thresholds in terms of mean and RMSE. The 95% credibility intervals (CI) of the various posteriors are also calculated and the lower and upper limits are referred to as CIL and CIU in the tables.

For the sake of comparison, the Hill ($\hat{\eta}_{Hill}$), generalised Hill ($\hat{\eta}_{Gen-Hill}$) and the biased reduced Hill ($\hat{\eta}_{BR-Hill}$) estimates are compared with the four Bayes estimates. Tables 1 and 2 show the dependence estimates for the four posteriors as well as the three classical estimates, when $n = 200$ and $n = 500$ observations are simulated from a bivariate normal distribution with $\rho = 0.9$ (thus, $\eta = 0.95$) respectively. The rest of the simulation results (from the bivariate normal and Morgenstern distributions with various parameter values and sample sizes) are shown in Tables B1 – B8 in Appendix B.1, found in the supplementary material to the paper.

The true parameter values can be found in [10]. For the bivariate Normal distribution $\eta = (1 + \rho)/2$. For the Morgenstern distribution, $\eta = 0.5$ for all values of α.

Tables 1 and 2 (where the tail dependence is large, close the one) show that $\hat{\eta}_{Beta}$ slightly outperforms the other Bayes and classical estimates over all four threshold values in terms of RMSE. The estimate $\hat{\eta}_{Beta}$ is also the closest to the true parameter value and the CI is the only interval, from the Bayes estimates, that includes the true parameter value, especially at larger thresholds. $\hat{\eta}_{TL}$ is in second place (in terms of RMSE). The classical estimators are performing slightly worse than $\hat{\eta}_{Beta}$ and $\hat{\eta}_{TL}$, in terms of RMSE, but is improving (as expected) as the sample size increases.

Tables B1 – B9 show that the good performance of $\hat{\eta}_{Beta}$ is constant over all the different simulation scenarios. The other Bayes estimates are also performing reasonably over the different scenarios. It is worth noting that for moderate values of η ($\eta = 0.5$) the $\hat{\eta}_{TEXP}$ performs well.

The sensitivity of the hyperparameters of the priors given in Section 3, were also considered. The value of the exponential hyperparameter was changed to 0.1 and 0.000001 respectively. Although the results are not presented in the paper, the conclusion was that the different values of the hyperparameter did not have a substantial influence on the estimates of η.

5 Properties of η when a beta prior is considered

The simulation study in Section 4 showed that $\hat{\eta}_{Beta}$ performed well in estimating the dependence parameter, for both large and moderate values of η. $\hat{\eta}_{Beta}$ presented a stable performance over different threshold values and for both sample sizes ($n = 200$ and $n = 500$). This is an important aspect in extreme value theory (EVT). Since the threshold is mostly unknown one would consider choosing the estimate that is less effected by the "wrong" threshold choice. Thus, choosing an estimate that remains stable over different thresholds, see for example [8] for more information.

In this section we derive some properties of η_{Beta}. When we assume that a and b are fixed, the conditional posterior of $\eta_{Beta}|a, b, \mathbf{t}$ from Equation (5) is proportional to

$$\pi_{Beta}(\eta|a, b, \mathbf{t}) \propto \eta^{a-N-1}(1-\eta)^{b-1}\mathrm{e}^{-1/\eta \sum_{i=1}^{N} \log t_i}, 0 < \eta < 1. \tag{9}$$

When (9) is multiplied by C, given in (10), where $\Gamma(\alpha, \beta)$ is an upper incomplete Gamma function, the conditional posterior (11) will be proper:

$$C = \left[S^{a-N}\left(\Gamma\left[N-a, S\right] - (b-1)\left(S\right)^{a-N+1}\Gamma\left[N-a-1, S\right]\right)\right]^{-1}, \tag{10}$$

with $S = \sum \log(t_i)$.

$$\pi_{Beta}(\eta|a, b, \mathbf{t}) = C\eta^{a-N-1}(1-\eta)^{b-1}\mathrm{e}^{-S/\eta}, 0 < \eta < 1, \tag{11}$$

see Appendix A.1.

Showing that the conditional posterior (11) is proper allows us to derive properties such as the mean, mode, and variance of $\eta|a, b, \mathbf{t}$.

The Bayes estimate of η, under squared error loss, is the mean of the posterior. The mean is

$$\begin{aligned} E(\eta_{Beta}|a, b, \mathbf{t}) &= C\int_0^1 \eta^{a-N}(1-\eta)^{b-1}\mathrm{e}^{-1/\eta S}\mathrm{d}\eta \\ &\approx CS^{1+a-N}(\Gamma[N-a-1, S] - (b-1)S^{2+a-N}\Gamma[N-a-2, S]). \end{aligned} \tag{12}$$

The variance of $\hat{\eta}_{Beta}$ is

$$Var[\eta_{Beta}|a, b, \mathbf{t}] = E[\eta^2_{Beta}|a, b, \mathbf{t}] - (E[\eta_{Beta}|a, b, \mathbf{t}])^2, \tag{13}$$

where

$$E(\eta^2|a, b, \mathbf{t}) \approx C[S^{2+a-N}(\Gamma[N-a-2, S] - (b-1)S^{3+a-N}\Gamma[N-a-3, S])].$$

Threshold	70^{th} percentile				50^{th} percentile				20^{th} percentile				10^{th} percentile			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Uni}$	0,85	0,10	0,789	0,892	0,86	0,09	0,811	0,888	0,85	0,10	0,818	0,878	0,85	0,11	0,819	0,872
$\hat{\eta}_{Beta}$	0,94	0,02	0,907	0,961	0,94	0,02	0,898	0,955	0,92	0,04	0,885	0,943	0,91	0,04	0,876	0,937
$\hat{\eta}_{TL}$	0,88	0,08	0,846	0,899	0,88	0,07	0,849	0,894	0,87	0,08	0,847	0,886	0,86	0,09	0,844	0,882
$\hat{\eta}_{TEXP}$	0,84	0,12	0,775	0,883	0,85	0,10	0,799	0,882	0,84	0,11	0,811	0,873	0,84	0,11	0,813	0,865
$\hat{\eta}_{Hill}$	0,86	0,10	0,772	0,939	0,86	0,09	0,798	0,904	0,84	0,11	0,808	0,877	0,84	0,11	0,808	0,869
$\hat{\eta}_{BR-Hill}$	0,83	0,16	0,525	0,980	0,86	0,12	0,667	0,985	0,87	0,10	0,793	0,977	0,87	0,09	0,803	0,989
$\hat{\eta}_{Gen-Hill}$	0,74	0,22	0,587	0,832	0,78	0,18	0,672	0,862	0,80	0,15	0,723	0,865	0,81	0,15	0,732	0,865

Table 1: $n = 200$ observations simulated from the bivariate normal with $\rho = 0.9$ (thus, $\eta = 0.95$).

Chosen	70^{th} percentile				50^{th} percentile				20^{th} percentile				10^{th} percentile			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Uni}$	0,87	0,08	0,829	0,912	0,87	0,08	0,838	0,9	0,86	0,09	0,835	0,879	0,85	0,10	0,828	0,871
$\hat{\eta}_{Beta}$	0,94	0,02	0,897	0,962	0,92	0,03	0,886	0,95	0,89	0,06	0,859	0,917	0,87	0,08	0,846	0,905
$\hat{\eta}_{TL}$	0,88	0,07	0,853	0,908	0,88	0,07	0,857	0,901	0,87	0,08	0,849	0,884	0,86	0,09	0,841	0,879
$\hat{\eta}_{TEXP}$	0,87	0,09	0,82	0,905	0,87	0,09	0,832	0,895	0,85	0,10	0,832	0,877	0,85	0,11	0,826	0,869
$\hat{\eta}_{Hill}$	0,88	0,08	0,820	0,932	0,87	0,09	0,829	0,9	0,85	0,10	0,831	0,875	0,84	0,11	0,825	0,868
$\hat{\eta}_{BR-Hill}$	0,87	0,11	0,718	0,955	0,88	0,08	0,814	0,977	0,88	0,08	0,837	0,984	0,88	0,08	0,837	0,986
$\hat{\eta}_{Gen-Hill}$	0,81	0,14	0,734	0,888	0,83	0,12	0,774	0,892	0,84	0,11	0,798	0,887	0,84	0,11	0,803	0,885

Table 2: $n = 500$ observations simulated from the bivariate normal with $\rho = 0.9$ (thus, $\eta = 0.95$).

Due to the incomplete gamma functions, (12) and (13) are restricted to relatively small values of N for reliable answers. In the case where N is too large, approximations for the incomplete gamma functions should be considered, see for example the "ZipfR" package in R, https://cran.r-project.org/web/packages/zipfR/zipfR.pdf, with the log scale setting as TRUE.

The Bayes estimate of η under absolute error loss is the mode of the posterior. The mode is derived in Appendix A.2 as

$$Mode(\eta_{Beta}|a,b,\mathbf{t}) = \frac{(a-1-N-S)+((N+1-a+S)^2+4(a-2+b-N)S)^{1/2}}{2(a-2+b-N)}. \tag{14}$$

It is beneficial in practice to express the estimates of η_{Beta} in terms of explicit formulas (even if it is only approximate). A simulation study is not always feasible and can be time consuming. A practitioner will prefer a formula to calculate the dependence parameter (see real data examples in Section 6).

Table 4, as well as Tables B9 – B32 in Appendix B.2, show how well the approximated mean and mode, given respectively by $\hat{\eta}_{Mean}$ and $\hat{\eta}_{Mode}$, of the posterior $\pi_{Beta}(\eta|a,b,\mathbf{t})$ perform. Table 4 shows the estimates, RMSE and 95% CI of $\hat{\eta}_{Beta}$, $\hat{\eta}_{Mean}$ and $\hat{\eta}_{Mode}$ when $n = 200$ observations are simulated from a bivariate normal distribution with $\rho = 0.9$ ($\eta = 0.95$), $a = 2.24$ and $b = 0.96$. The simulations are repeated 100 times and the mean, over the repetitions, is taken as the estimates. At lower thresholds (20th and 10th percentiles) $\hat{\eta}_{Mean}$ results in NAN values (indicating that the numerical computations broke down) due to the large N in the incomplete gamma functions (from (12)). Further investigation should be done in terms of approximating the incomplete gamma functions. At higher thresholds (the 50th and 70th percentiles) $\hat{\eta}_{Mean}$ is close to $\hat{\eta}_{Beta}$ with small RMSEs. $\hat{\eta}_{Mode}$ performs relatively well (small RMSE) over all the different threshold levels.

Tables B9 – B32 show the results for $n = 200$ observations simulated from the Bivariate normal with $\rho = 0.9$ ($\eta = 0.95$) and $\rho = 0.6$ ($\eta = 0.8$). The effects of a and b are also tested and shown in these tables. Different scenarios (of a and b) are considered, see Table 3. Since the prior is assumed to follow a beta distribution with parameters a and b, different means and variances for η can be chosen, a and b are then calculated accordingly. For the problem to be meaningful we should have that $Var(\eta) < E(\eta)(1 - E(\eta))$.

Tables B9 – B30 show that at lower thresholds (20th and 10th percentiles) $\hat{\eta}_{Mean}$ results in NAN or unreliable results (large RMSEs) due to the large N in the incomplete gamma functions. At higher thresholds (the 50th and 70th percentiles) and $\rho = 0.6$, $\hat{\eta}_{Mean}$ is close to $\hat{\eta}_{Mean}$ with small RMSEs, regardless of the values of a and b. For a larger value of $\rho = 0.9$ the RMSEs of $\hat{\eta}_{Mean}$ is slightly larger (than in the $\rho = 0.6$ case) but is still performing relatively well, regardless of the values of a and b. Therefore, different values of a and b have no effect on how well the estimates perform.

For higher thresholds (50th and 70th percentiles) and a high $\rho = 0.9$, $\hat{\eta}_{Mode}$, for small values of a and b, mostly does not exist (see Tables B9 – B14), since the square root in (14) cannot be negative. Care should be taken when choosing a and b (especially at large thresholds). From the 100 repetitions, around 80%, resulted in taking the square root of a negative value. In the simulations where less than 10% of the repetitions resulted in negative values (for the square root) the results were obtained from the more than 90% that resulted in valid answers. At smaller thresholds (the 20th and 10th percentiles) and $\rho = 0.9$, $\hat{\eta}_{Mode}$ is close to $\hat{\eta}_{Beta}$ with small RMSEs, regardless of the values of a and b. For $\rho = 0.6$ the RMSEs of $\hat{\eta}_{Mode}$ is performing well, regardless of the values of a and b. Similar outcomes were obtained when the Morgenstern distribution was considered. The tables are omitted from this paper (due to space). Thus, for appropriate values of a, b and N, $\hat{\eta}_{Mean}$ and $\hat{\eta}_{Mode}$ are appropriate approximations for $\hat{\eta}_{Beta}$.

The cumulative distribution function (CDF) of η can be derived from (11) as:

$$\begin{aligned} F(\eta|a,b,\mathbf{t}) &= C\int_0^{\eta} \eta^{a-N-1}(1-\eta)^{b-1}\mathrm{e}^{-S/\eta}\mathrm{d}\eta \\ &= C[S^{a-N}(\Gamma[N-a,S/\eta]-(b-1)S^{a-N+1}\Gamma[N-a-1,S/\eta])], \end{aligned} \quad (15)$$

by using the same approximations as in Appendix A.1. In the case where N is large, approximations for the incomplete gamma functions should be considered. Having an explicit formula for the CDF is handy, especially when estimating probabilities (see Section 6).

6 Real life examples

Two data sets are considered in this section.

6.1 Bloemfontein rainfall data

The first dataset that we consider is Bloemfontein rainfall data. Bloemfontein is the capital city of the Free State province in South Africa and is a summer rainfall region. In this example we consider a small dataset that involves the total rainfall of two of the summer months, January and February for the years 1970 to 2017. The aim is to model the dependence between the total rainfall of the two months.

	$Var(\eta) = 0.2$	$Var(\eta) = 0.1$	$Var(\eta) = 0.07$	$Var(\eta) = 0.03$
$E(\eta) = 0.7$	$a = 0.035, b = 0.015$	$a = 0.77, b = 0.33$	$a = 1.4, b = 0.6$	$a = 4.2, b = 1.8$
$E(\eta) = 0.5$	$a = 0.125, b = 0.125$	$a = 0.75, b = 0.75$	$a = 1.286, b = 1.286$	$a = 3.67, b = 3.67$
$E(\eta) = 0.3$	$a = 0.015, b = 0.035$	$a = 0.33, b = 0.77$	$a = 0.6, b = 1.4$	$a = 1.8, b = 4.2$

Table 3: Different scenarios for choosing a and b.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,907	0,961	0,94	0,02	0,898	0,955	0,92	0,04	0,885	0,943	0,91	0,04	0,876	0,937
$\hat{\eta}_{Mean}$	0.867	0.09	0.822	0.91	0.863	0.09	0.816	0.898	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0.887	0.074	0.802	0.949	0.869	0.088	0.806	0.925	0.856	0.1	0.822	0.886	0.847	0.104	0.814	0.875

Table 4: $n = 200$ observations simulated from the bivariate normal with $\rho = 0.9$ ($\eta = 0.95$), $a = 2.24$ and $b = 0.96$.

Figure 1 shows a scatterplot of the two variables; rainfall in January and rainfall in February. From the scatterplot it is evident that there is dependence between the two random variables. Figure 2 shows the dependance estimates $\hat{\eta}_{Mean}$, $\hat{\eta}_{Mode}$ and $\hat{\eta}_{Hill}$ over a range of thresholds (indicated by k, where k represents the number of observations above the threshold). Since Figure 1 shows a rather high dependance between the two variables, the mean and variance of η were chosen (as prior information) as $E(\eta) = 0.7$ and $Var(\eta) = 0.03$ with $a = 4.2$ and $b = 1.8$. From Figure 2 the two estimates ($\hat{\eta}_{Mode}$ and $\hat{\eta}_{Mean}$) are close to one another, and they present a stable behaviour (of $\hat{\eta}$) over the different thresholds. The Hill estimate however shows a more unstable behaviour over the different thresholds. This is expected from the Hill estimate, which is known for its bias, especially in small sample sizes. The probabilities for N not too large can now easily be calculated using (15). For example, if a threshold is chosen at $k = 25$, thus only considering the top sorted 25 observations, $\hat{\eta}_{Mode} = 0.802$. The probability that the random variables are asymptotically independent, $P(\eta < 1) = 1$, is 100% the $P(\eta < 0.5) = 0.001$ and $P(\eta \geq 0.7) = 1 - 0.1492 = 0.8508$ etc.

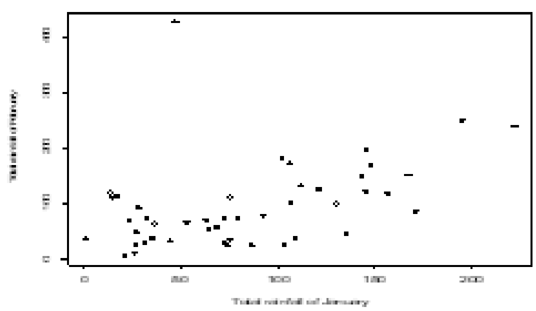

Figure 1: Total rainfall of January versus total rainfall of February.

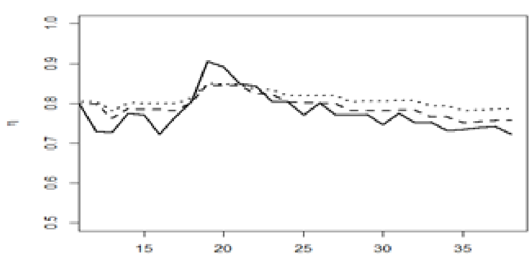

Figure 2: $\hat{\eta}_{Mean}$ is represented by the dotted line, $\hat{\eta}_{Mode}$ by the dashed line and $\hat{\eta}_{Hill}$ by the solid line.

6.2 Condroz data

This data frame, consisting of 1505 observations, can be found in the "robustbase" R package, see https://cran.r-project.org/web/packages/robustbase/robustbase.pdf. Each observation contains a Calcium content and pH level measurement of soil samples of the Condroz region (the Southern part of Belgium). Refer to [7] for more information on the dataset. We want to estimate the dependence parameter for these two variables. Beirlant [3] also considered this dataset but not for the purpose of estimating the dependence parameter. A scatterplot of the data is given in Figure 3. There seems to be some dependence between the two variables, the pH increases as the Calcium content increases. Figure 4 shows the dependance estimates $\hat{\eta}_{Mode}$ and $\hat{\eta}_{Hill}$ over a large range of thresholds. $\hat{\eta}_{Mean}$ is not considered in Figure 4, because the sample size is too large (and the incomplete gamma functions are problematic). Again, the mean and variance of η were chosen (as prior information) as $E(\eta) = 0.7$ and $Var(\eta) = 0.03$ with $a = 4.2$ and $b = 1.8$.

From Figure 4 it can be seen that for a large number of observations above the threshold (large k) the two estimates perform similarly. Figure 5 however shows that when we zoom into the plot, only looking at the large thresholds (small number of observations above the threshold) $\hat{\eta}_{Mean}$ and $\hat{\eta}_{Mode}$ show more stable and constant results than $\hat{\eta}_{Hill}$. Often in EVT one is faced with a small number of observations. It is therefore noteworthy that the performances of $\hat{\eta}_{Mode}$ and $\hat{\eta}_{Mean}$ are stable for a small number of observations.

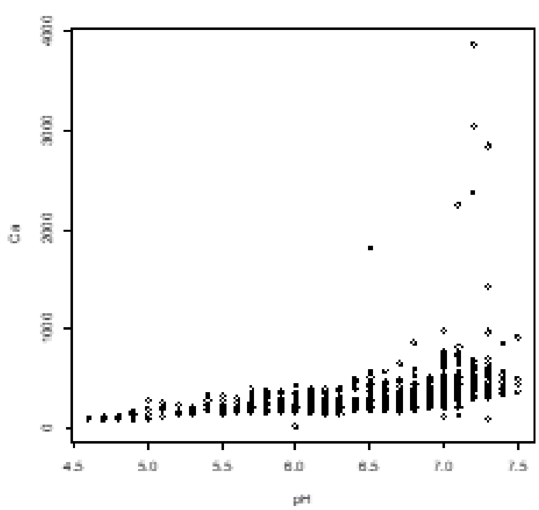

Figure 3: pH level measurements versus Calcium content.

7 Conclusion

In this study we have shown that the dependence parameter, in a bivariate extreme value setting, following the method of [10], can effectively be estimated through a Bayesian approach when a prior, restricted between 0 and 1 is assumed. The Bayesian approach gives the advantage of incorporating and enforcing the prior restriction $\eta \in (0, 1]$. The estimates of η, obtained through the simulations from the various posterior distributions, gave sensible results (as seen in the simulation study). Although all four Bayes estimates performed well, the estimate $\hat{\eta}_{Beta}$ was further investigated because it showed a stable

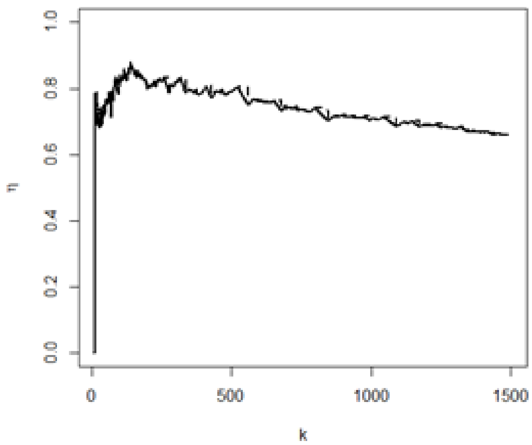

Figure 4: $\hat{\eta}_{Mode}$ is represented by the dashed line and $\hat{\eta}_{Hill}$ by the solid line.

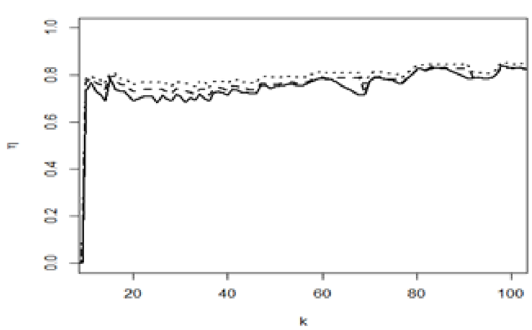

Figure 5: $\hat{\eta}_{Mean}$ is represented by the dotted, $\hat{\eta}_{Mode}$ by the dashed line and $\hat{\eta}_{Hill}$ by the solid line.

performance throughout the simulation study, for both large and moderate values of η. It was further proven that the conditional posterior, $\eta_{Beta}|a, b, t$, is proper and two theoretical approximate estimates of η, the mean and the mode, were derived. These theoretical formulas (given in (12) and (14)) simplify the estimation process in practical applications, as illustrated in the Section 6. The practical applications in Section 6 showed that for large threshold values (small number of observations) $\hat{\eta}_{Beta}$ is stable (also when compared to the Hill estimate). Small sample sizes are often a dilemma in EVT. Probabilities of η can also be calculated from the CDF in (15). Further investigation should be done on approximating the incomplete gamma functions for large sample sizes.

Appendix A.1: Showing that the conditional posterior $\pi_{Beta}(\eta|a, b, \mathbf{t})$ is proper.

As was done in the main text, let $S = \sum_{i=1}^{N} \log t_i$. Equation (9) gives the conditional posterior $\eta|a, b, t$ when a beta prior is assumed

$$\pi_{Beta}(\eta|a, b, \mathbf{t}) \propto \eta^{a-N-1}(1-\eta)^{b-1}\mathrm{e}^{-S/\eta}, 0 < \eta < 1.$$

When this posterior is integrated over its domain it integrates to a constant as follows:

$$\begin{aligned}\int_0^1 \eta^{a-N-1}(1-\eta)^{b-1}\mathrm{e}^{-S/\eta}\mathrm{d}\eta &= \int_0^1 \eta^{a-N-1}\mathrm{e}^{-S/\eta}\mathrm{d}\eta - (b-1)\int_0^1 \eta^{a-N}\mathrm{e}^{-S/\eta}\mathrm{d}\eta \\ &= S^{a-N}(\Gamma[N-a, S] - (b-1)S^{a-N+1}\Gamma[N-a-1, S]).\end{aligned}$$

This follows after using the Maclaurin and Taylor approximations: $\log(1-\eta) \approx -\eta$ and $\mathrm{e}^{-\eta(b-1)} \approx [1-\eta(b-1)]$. Since the posterior integrates to a constant the posterior is regarded as proper.

Appendix A.2: Deriving the mode of the conditional posterior $\pi_{Beta}(\eta|a, b, \mathbf{t})$

Consider the conditional posterior from (11). The logarithm of the posterior is:

$$\log(\pi_{Beta}(\eta|a, b, \mathbf{t})) = \log C + (a-N-1)\log\eta + (b-1)\log(1-\eta) - S/\eta.$$

The derivative with respect to η set equal to zero is:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}\log\pi_{Beta}(\eta|a, b, \mathbf{t}) = \frac{a-1-N}{\eta} - \frac{b-1}{1-\eta} + \frac{S}{\eta^2} &= 0 \\ \iff \eta^2(a-2+b-N) - \eta(a-1-N-S) - S &= 0.\end{aligned}$$

This quadratic equation can be solved as:

$$\eta = \frac{a-1-N-S+((N+1-a+S)^2+4S(a-2+b-N))^{1/2}}{2(a-2+b-N)}.$$

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Supplementary Material to "Estimating the dependence parameter in bivariate extreme value statistics through a Bayesian approach"

Appendix B.1: Additional tables for the simulation study.

Table B1: $n = 500$ observations simulated from the Bivariate Normal with $\rho = 0.6$ (thus, $\eta = 0.8$).

Chosen	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}
$\hat{\eta}_{Uni}$	0,73	0,08	0,6611	0,8135	0,72	0,08	0,6668	0,7660	0,70	0,10	0,6696	0,7318	0,69	0,11	0,6671	0,7196
$\hat{\eta}_{Beta}$	0,77	0,06	0,6741	0,8813	0,74	0,07	0,6723	0,7871	0,71	0,10	0,6727	0,7385	0,70	0,10	0,6694	0,7243
$\hat{\eta}_{TL}$	0,77	0,05	0,6964	0,8433	0,75	0,06	0,6827	0,7942	0,71	0,09	0,6790	0,7475	0,70	0,10	0,6773	0,7323
$\hat{\eta}_{TEXP}$	0,73	0,08	0,6559	0,8077	0,72	0,09	0,6619	0,7629	0,70	0,10	0,6668	0,7281	0,69	0,11	0,6650	0,7162
$\hat{\eta}_{Hill}$	0,72	0,09	0,6537	0,8047	0,72	0,09	0,6603	0,7606	0,70	0,10	0,6646	0,7280	0,69	0,11	0,6635	0,7146
$\hat{\eta}_{BR-Hill}$	0,74	0,12	0,5100	0,9375	0,74	0,08	0,6511	0,8934	0,75	0,08	0,6575	0,8607	0,75	0,08	0,6608	0,8601
$\hat{\eta}_{Gen-Hill}$	0,69	0,13	0,5418	0,7920	0,70	0,11	0,5994	0,7769	0,71	0,10	0,6299	0,7691	0,71	0,10	0,6338	0,7646

Table B2: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.6$ (thus, $\eta = 0.8$).

Chosen	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}
$\hat{\eta}_{Uni}$	0,73	0,09	0,6309	0,8399	0,72	0,09	0,6454	0,8018	0,70	0,11	0,6435	0,7580	0,69	0,11	0,6449	0,7409
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{TL}$	0,80	0,04	0,7040	0,8690	0,77	0,05	0,6924	0,8430	0,73	0,07	0,6722	0,7951	0,72	0,08	0,6669	0,7764
$\hat{\eta}_{TEXP}$	0,71	0,11	0,6088	0,8263	0,71	0,10	0,6356	0,7907	0,69	0,11	0,6371	0,7514	0,68	0,12	0,6390	0,7331
$\hat{\eta}_{Hill}$	0,71	0,11	0,6057	0,8356	0,70	0,11	0,6311	0,7893	0,69	0,12	0,6327	0,7470	0,68	0,12	0,6377	0,7292
$\hat{\eta}_{BR-Hill}$	0,72	0,17	0,4700	0,9853	0,73	0,12	0,5600	0,9200	0,73	0,11	0,6086	0,9609	0,73	0,11	0,6221	0,9580
$\hat{\eta}_{Gen-Hill}$	0,62	0,21	0,3866	0,7958	0,64	0,17	0,4800	0,7865	0,66	0,15	0,5489	0,7840	0,67	0,14	0,5624	0,7804

Table B3: $n = 200$ observations simulated from the Morgenstern's Distribution with $\alpha = 0$ (thus, $\eta = 0.5$).

Chosen	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}
$\hat{\eta}_{Uni}$	0,51	0,05	0,4238	0,6219	0,51	0,04	0,4391	0,5793	0,50	0,03	0,4601	0,5546	0,50	0,02	0,4652	0,5456
$\hat{\eta}_{Beta}$	0,52	0,06	0,4222	0,6580	0,51	0,04	0,4374	0,5848	0,50	0,03	0,4596	0,5552	0,50	0,02	0,4657	0,5461
$\hat{\eta}_{TL}$	0,55	0,09	0,4464	0,6981	0,53	0,05	0,4509	0,6082	0,51	0,03	0,4673	0,5679	0,51	0,03	0,4736	0,5571
$\hat{\eta}_{TEXP}$	0,50	0,05	0,4127	0,6021	0,50	0,04	0,4322	0,5701	0,50	0,02	0,4547	0,5480	0,50	0,02	0,4611	0,5407
$\hat{\eta}_{Hill}$	0,49	0,05	0,4081	0,5906	0,50	0,04	0,4306	0,5671	0,50	0,02	0,4526	0,5448	0,50	0,02	0,4599	0,5388
$\hat{\eta}_{BR-Hill}$	0,48	0,14	0,0959	0,7332	0,48	0,11	0,2445	0,7110	0,49	0,09	0,3029	0,6423	0,49	0,09	0,3227	0,6440
$\hat{\eta}_{Gen-Hill}$	0,41	0,14	0,1663	0,6015	0,43	0,11	0,2615	0,6019	0,45	0,08	0,3145	0,5795	0,46	0,08	0,3328	0,5790

Table B4: $n = 200$ observations simulated from the Morgenstern's Distribution with $\alpha = -0.75$ (thus, $\eta = 0.5$).

Chosen	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}
$\hat{\eta}_{Uni}$	0,43	0,09	0,3639	0,5122	0,43	0,08	0,3736	0,4878	0,44	0,07	0,3999	0,4857	0,44	0,06	0,4112	0,4848
$\hat{\eta}_{Beta}$	0,42	0,09	0,3556	0,5156	0,42	0,08	0,3707	0,4871	0,43	0,07	0,3989	0,4869	0,44	0,06	0,4088	0,4856
$\hat{\eta}_{TL}$	0,45	0,07	0,3733	0,5522	0,44	0,07	0,3799	0,5037	0,44	0,06	0,4049	0,4971	0,45	0,06	0,4164	0,4939
$\hat{\eta}_{TEXP}$	0,41	0,10	0,3508	0,4990	0,42	0,09	0,3669	0,4786	0,43	0,07	0,3952	0,4814	0,44	0,07	0,4059	0,4809
$\hat{\eta}_{Hill}$	0,41	0,10	0,3495	0,4959	0,42	0,09	0,3632	0,4726	0,43	0,07	0,3935	0,4809	0,44	0,07	0,4060	0,4787
$\hat{\eta}_{BR-Hill}$	0,39	0,16	0,1369	0,6053	0,38	0,15	0,2070	0,5220	0,38	0,14	0,2360	0,5022	0,38	0,13	0,2447	0,5010
$\hat{\eta}_{Gen-Hill}$	0,34	0,20	0,1061	0,5352	0,36	0,17	0,1840	0,5080	0,36	0,15	0,2282	0,4836	0,36	0,15	0,2386	0,4797

Table B5: $n = 200$ observations simulated from the Morgenstern's Distribution with $\alpha = 0.75$ (thus, $\eta = 0.5$).

Chosen	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}
$\hat{\eta}_{Uni}$	0,57	0,09	0,4670	0,6785	0,59	0,09	0,5047	0,6732	0,58	0,08	0,5329	0,6347	0,57	0,08	0,5274	0,6214
$\hat{\eta}_{Beta}$	0,59	0,12	0,4672	0,7484	0,59	0,10	0,5050	0,7006	0,58	0,09	0,5338	0,6419	0,58	0,08	0,5301	0,6253
$\hat{\eta}_{TL}$	0,63	0,15	0,4955	0,7604	0,62	0,13	0,5247	0,7280	0,60	0,10	0,5443	0,6609	0,59	0,09	0,5390	0,6406
$\hat{\eta}_{TEXP}$	0,56	0,08	0,4556	0,6628	0,58	0,09	0,4958	0,6620	0,57	0,08	0,5269	0,6282	0,57	0,07	0,5246	0,6138
$\hat{\eta}_{Hill}$	0,55	0,08	0,4500	0,6559	0,57	0,08	0,4949	0,6558	0,57	0,07	0,5243	0,6247	0,56	0,07	0,5228	0,6133
$\hat{\eta}_{BR-Hill}$	0,50	0,15	0,1229	0,7487	0,52	0,13	0,1686	0,6776	0,54	0,11	0,2694	0,6776	0,55	0,11	0,3086	0,6973
$\hat{\eta}_{Gen-Hill}$	0,43	0,13	0,1645	0,6173	0,47	0,10	0,2512	0,6096	0,50	0,07	0,3526	0,6277	0,51	0,07	0,3741	0,6298

Table B6: $n = 500$ observations simulated from the Morgenstern's Distribution with $\alpha = 0$ (thus, $\eta = 0.5$).

Chosen	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}
$\hat{\eta}_{Uni}$	0,51	0,03	0,4484	0,5759	0,51	0,02	0,4561	0,5527	0,50	0,02	0,4561	0,5527	0,50	0,02	0,4778	0,5337
$\hat{\eta}_{Beta}$	0,51	0,04	0,4470	0,5809	0,51	0,02	0,4558	0,5540	0,50	0,02	0,4558	0,5540	0,50	0,02	0,4771	0,5338
$\hat{\eta}_{TL}$	0,52	0,04	0,4569	0,5948	0,51	0,03	0,4615	0,5621	0,51	0,02	0,4615	0,5621	0,51	0,02	0,4806	0,5373
$\hat{\eta}_{TEXP}$	0,51	0,03	0,4435	0,5706	0,50	0,02	0,4528	0,5489	0,50	0,02	0,4528	0,5489	0,50	0,01	0,4751	0,5318
$\hat{\eta}_{Hill}$	0,50	0,03	0,4414	0,5693	0,50	0,02	0,4526	0,5470	0,50	0,02	0,4526	0,5470	0,50	0,01	0,4745	0,5308
$\hat{\eta}_{BR-Hill}$	0,50	0,10	0,2983	0,6833	0,50	0,07	0,3493	0,6438	0,50	0,06	0,3493	0,6438	0,50	0,05	0,3853	0,6088
$\hat{\eta}_{Gen-Hill}$	0,46	0,09	0,3226	0,6052	0,48	0,07	0,3603	0,5864	0,48	0,05	0,3603	0,5864	0,49	0,05	0,3959	0,5647

Table B7: $n = 500$ observations simulated from the Morgenstern's Distribution with $\alpha = -0.75$ (thus, $\eta = 0.5$)

Chosen	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}
$\hat{\eta}_{Uni}$	0,41	0,10	0,3522	0,4694	0,42	0,09	0,3827	0,4489	0,43	0,07	0,4045	0,4537	0,43	0,07	0,4121	0,4559
$\hat{\eta}_{Beta}$	0,41	0,10	0,3502	0,4685	0,41	0,09	0,3817	0,4499	0,43	0,07	0,4045	0,4532	0,43	0,07	0,4115	0,4554
$\hat{\eta}_{TL}$	0,42	0,09	0,3563	0,4803	0,42	0,08	0,3862	0,4543	0,43	0,07	0,4065	0,4559	0,44	0,07	0,4140	0,4582
$\hat{\eta}_{TEXP}$	0,40	0,10	0,3483	0,4633	0,41	0,09	0,3808	0,4464	0,43	0,07	0,4030	0,4517	0,43	0,07	0,4105	0,4542
$\hat{\eta}_{Hill}$	0,40	0,10	0,3467	0,4614	0,41	0,09	0,3791	0,4457	0,43	0,08	0,4024	0,4500	0,43	0,07	0,4097	0,4536
$\hat{\eta}_{BR-Hill}$	0,39	0,13	0,2448	0,5734	0,39	0,13	0,2612	0,4723	0,38	0,12	0,2823	0,4458	0,38	0,12	0,2856	0,4464
$\hat{\eta}_{Gen-Hill}$	0,37	0,15	0,2149	0,5196	0,37	0,14	0,2491	0,4803	0,37	0,14	0,2798	0,4656	0,37	0,14	0,2867	0,4593

Table B8: $n = 500$ observations simulated from the Morgenstern's Distribution with $\alpha = 0.75$ (thus, $\eta = 0.5$).

Chosen	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}	Mean	RMSE	CI_{Lower}	CI_{Upper}
$\hat{\eta}_{Uni}$	0,57	0,08	0,5127	0,6318	0,58	0,09	0,5327	0,6316	0,58	0,08	0,5389	0,6083	0,57	0,07	0,5409	0,6016
$\hat{\eta}_{Beta}$	0,57	0,08	0,5126	0,6391	0,58	0,09	0,5317	0,6354	0,58	0,08	0,5391	0,6104	0,57	0,07	0,5428	0,6024
$\hat{\eta}_{TL}$	0,59	0,10	0,5256	0,6571	0,59	0,10	0,5391	0,6465	0,58	0,08	0,5437	0,6166	0,58	0,08	0,5464	0,6070
$\hat{\eta}_{TEXP}$	0,56	0,07	0,5071	0,6229	0,58	0,08	0,5283	0,6263	0,57	0,08	0,5376	0,6057	0,57	0,07	0,5398	0,5983
$\hat{\eta}_{Hill}$	0,56	0,07	0,5049	0,6223	0,58	0,08	0,5266	0,6264	0,57	0,07	0,5351	0,6038	0,57	0,07	0,5387	0,5978
$\hat{\eta}_{BR-Hill}$	0,51	0,10	0,2772	0,6724	0,54	0,09	0,3456	0,6385	0,57	0,09	0,4224	0,6315	0,57	0,09	0,4496	0,6338
$\hat{\eta}_{Gen-Hill}$	0,49	0,07	0,3292	0,6338	0,51	0,06	0,3933	0,6042	0,54	0,06	0,4487	0,6049	0,54	0,06	0,4603	0,6056

Appendix B.2: Additional tables for the approximation of $\hat{\eta}_{Beta}$.

Table B9: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 0.035$ and $b = 0.015$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,85	0,11	0,7805	0,8966	0,86	0,10	0,8003	0,8976	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN	0,87	0,08	0,8286	0,9081	0,87	0,09	0,8210	0,9110

Table B10: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 0.125$ and $b = 0.125$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,85	0,10	0,7867	0,8991	0,86	0,10	0,8032	0,8928	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN	0,87	0,08	0,8257	0,9082	0,86	0,09	0,8210	0,9130

Table B11: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 0.015$ and $b = 0.035$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,85	0,11	0,7817	0,8953	0,86	0,10	0,8089	0,8904	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN	0,87	0,09	0,8213	0,9175	0,86	0,09	0,8158	0,9071

Table B12: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 0.77$ and $b = 0.33$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,85	0,10	0,7832	0,8977	0,86	0,09	0,8236	0,8966	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN	0,87	0,08	0,8162	0,9184	0,86	0,09	0,8145	0,8968

Table B13: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 0.75$ and $b = 0.75$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,86	0,10	0,8027	0,8959	0,86	0,09	0,8263	0,8963	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	NAN	NAN	NAN	NAN	0,87	0,08	0,8250	0,9249	0,86	0,10	0,8182	0,8927	0,85	0,10	0,8130	0,8737

Table B14: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 0.33$ and $b = 0.77$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,85	0,10	0,7985	0,8938	0,86	0,10	0,8042	0,8881	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	NAN	NAN	NAN	NAN	0,87	0,09	0,7977	0,9161	0,85	0,10	0,8134	0,8809	0,84	0,11	0,8070	0,8713

Table B15: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 1.4$ and $b = 0.6$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,86	0,10	0,7990	0,8969	0,86	0,09	0,8161	0,9019	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,85	0,10	0,7814	0,8958	0,88	0,08	0,8158	0,9208	0,86	0,09	0,8231	0,8971	0,86	0,09	0,8242	0,8902

Table B16: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 1.286$ and $b = 1.286$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,85	0,10	0,7921	0,8929	0,86	0,09	0,8114	0,8988	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,84	0,11	0,7691	0,9058	0,85	0,10	0,7938	0,9021	0,84	0,11	0,8068	0,8668	0,84	0,11	0,8095	0,8647

Table B17: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 0.6$ and $b = 1.4$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,84	0,11	0,7829	0,8876	0,86	0,10	0,8052	0,8947	NAN	NAN	NAN	NAN	NANN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,83	0,13	0,7569	0,8894	0,84	0,11	0,7856	0,8906	0,84	0,12	0,8013	0,8600	0,83	0,12	0,8045	0,8586

Table B18: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 4.2$ and $b = 1.8$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,86	0,09	0,8064	0,8978	0,87	0,09	0,8204	0,9022	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,85	0,11	0,7819	0,8922	0,85	0,10	0,8002	0,8925	0,84	0,11	0,8103	0,8649	0,84	0,11	0,8125	0,8632

Table B19: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 3.67$ and $b = 3.67$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,88	0,07	0,8382	0,9155	0,89	0,07	0,8431	0,9169	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,78	0,18	0,7249	0,8170	0,80	0,15	0,7574	0,8336	0,81	0,15	0,7786	0,8245	0,81	0,14	0,7833	0,8261

Table B20: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.9$ (thus, $\eta = 0.95$), $a = 1.8$ and $b = 4.8$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$	Mean	RMSE	CI$_L$	CI$_U$
$\hat{\eta}_{Beta}$	0,94	0,02	0,9074	0,9609	0,94	0,02	0,8984	0,9549	0,92	0,04	0,8853	0,9434	0,91	0,04	0,8764	0,9365
$\hat{\eta}_{Mean}$	0,87	0,08	0,8178	0,9059	0,87	0,08	0,8287	0,9090	NAN	NAN	NAN	NAN	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,75	0,20	0,6999	0,7923	0,78	0,17	0,7392	0,8147	0,79	0,16	0,7652	0,8102	0,79	0,16	0,7709	0,8128

Table B21: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.6$ (thus, $\eta = 0.8$), $a = 0.035$ and $b = 0.015$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,73	0,09	0,6246	0,8365	0,72	0,09	0,6517	0,7971	1,25	0,68	0,5369	1,7571	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,74	0,09	0,6154	0,8399	0,73	0,09	0,6479	0,8168	0,70	0,10	0,6511	0,7464	0,69	0,11	0,6469	0,7331

Table B22: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.6$ (thus, $\eta = 0.8$), $a = 0.125$ and $b = 0.125$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,73	0,08	0,6252	0,8371	0,72	0,09	0,6521	0,7975	1,26	0,70	0,4385	1,8998	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,74	0,09	0,6148	0,8498	0,72	0,09	0,6471	0,8125	0,70	0,10	0,6506	0,7453	0,69	0,11	0,6464	0,7322

Table B23: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.6$ (thus, $\eta = 0.8$), $a = 0.015$ and $b = 0.035$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,73	0,09	0,6243	0,8363	0,72	0,09	0,6515	0,7969	1,29	0,69	0,5547	1,7871	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,73	0,09	0,6148	0,8356	0,72	0,09	0,6475	0,8156	0,70	0,10	0,6508	0,7460	0,69	0,11	0,6466	0,7328

Table B24: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0.6$ (thus, $\eta = 0.8$), $a = 0.77$ and $b = 0.33$.

Threshold	70^{th} $percentile$				50^{th} $percentile$				20^{th} $percentile$				10^{th} $percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,74	0,08	0,6316	0,8423	0,73	0,08	0,6560	0,8018	1,41	1,02	0,4482	2,8948	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,74	0,09	0,6183	0,8368	0,72	0,09	0,6489	0,8101	0,70	0,10	0,6517	0,7455	0,69	0,11	0,6474	0,7325

Table B25: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0{,}6$ (thus, $\eta = 0{,}8$), $a\ =\ 0{,}75$ and $b = 0{,}75$,

Threshold	$70^{th}\ percentile$				$50^{th}\ percentile$				$20^{th}\ percentile$				$10^{th}\ percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,74	0,08	0,6296	0,8404	0,73	0,08	0,6548	0,8003	2,35	2,60	0,4687	5,8872	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,73	0,09	0,6112	0,8430	0,72	0,09	0,6437	0,7944	0,70	0,10	0,6484	0,7395	0,69	0,11	0,6446	0,7277

Table B26: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0{,}6$ (thus, $\eta = 0{,}8$), $a\ =\ 0{,}33$ and $b = 0{,}77$,

Threshold	$70^{th}\ percentile$				$50^{th}\ percentile$				$20^{th}\ percentile$				$10^{th}\ percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,73	0,08	0,6250	0,8366	0,72	0,09	0,6519	0,7972	2,64	2,79	0,6185	6,0221	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,72	0,10	0,6065	0,8345	0,71	0,10	0,6407	0,7902	0,70	0,11	0,6465	0,7373	0,69	0,11	0,6429	0,7258

Table B27: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0{,}6$ (thus, $\eta = 0{,}8$),$a\ =\ 1.4$ and $b = 0.6$.

Threshold	$70^{th}\ percentile$				$50^{th}\ percentile$				$20^{th}\ percentile$				$10^{th}\ percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,75	0,08	0,6375	0,8468	0,73	0,08	0,6596	0,8057	1,63	1,51	0,6166	3,9430	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,74	0,09	0,6206	0,8278	0,72	0,09	0,6498	0,8053	0,70	0,10	0,6522	0,7447	0,69	0,11	0,6479	0,7321

Table B28: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0{,}6$ (thus, $\eta = 0{,}8$),$a\ =\ 1.286$ and $b = 1.286$.

Threshold	$70^{th}\ percentile$				$50^{th}\ percentile$				$20^{th}\ percentile$				$10^{th}\ percentile$			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,74	0,08	0,6315	0,8409	0,73	0,08	0,6558	0,8009	-0,47	2,04	-3,157	0,8908	NAN	NA	NAN	NAN
$\hat{\eta}_{Mode}$	0,72	0,10	0,6082	0,8287	0,71	0,10	0,6409	0,7828	0,70	0,11	0,6466	0,7351	0,69	0,11	0,6430	0,7241

Table B29: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0{,}6$ (thus, $\eta = 0{,}8$), $a = 0.6$ and $b = 1.4$.

Threshold	70^{th} *percentile*				50^{th} *percentile*				20^{th} *percentile*				10^{th} *percentile*			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,73	0,09	0,6228	0,8331	0,72	0,09	0,6505	0,7946	-0,49	1,76	-1,980	0,7093	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,71	0,11	0,5997	0,8146	0,70	0,10	0,6354	0,7748	0,69	0,11	0,6430	0,7306	0,68	0,12	0,6399	0,7203

Table B30: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0{,}6$ (thus, $\eta = 0{,}8$), $a = 4.2$ and $b = 1.8$.

Threshold	70^{th} *percentile*				50^{th} *percentile*				20^{th} *percentile*				10^{th} *percentile*			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,76	0,07	0,6535	0,8508	0,74	0,07	0,6690	0,8104	0,70	0,16	0,6504	0,7918	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,73	0,08	0,6293	0,8328	0,72	0,09	0,6534	0,7902	0,70	0,10	0,6544	0,7416	0,69	0,11	0,6500	0,7302

Table B31: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0{,}6$ (thus, $\eta = 0{,}8$), $a = 3.67$ and $b = 3.67$.

Threshold	70^{th} *percentile*				50^{th} *percentile*				20^{th} *percentile*				10^{th} *percentile*			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,79	0,05	0,6888	0,8769	0,76	0,06	0,6899	0,8336	0,73	0,24	0,4234	0,7661	NANN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,68	0,12	0,5969	0,7662	0,69	0,12	0,6305	0,7489	0,68	0,12	0,6393	0,7188	0,68	0,12	0,6367	0,7107

Table B32: $n = 200$ observations simulated from the Bivariate Normal with $\rho = 0{,}6$ (thus, $\eta = 0{,}8$), $a = 1.8$ and $b = 4.2$.

Threshold	70^{th} *percentile*				50^{th} *percentile*				20^{th} *percentile*				10^{th} *percentile*			
Estimate	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U	Mean	RMSE	CI_L	CI_U
$\hat{\eta}_{Beta}$	0,82	0,08	0,6630	0,9385	0,77	0,07	0,6631	0,8904	0,72	0,09	0,6522	0,7922	0,70	0,10	0,6520	0,7668
$\hat{\eta}_{Mean}$	0,77	0,06	0,6618	0,8614	0,74	0,07	0,6739	0,8187	0,64	0,27	0,2756	1,0239	NAN	NAN	NAN	NAN
$\hat{\eta}_{Mode}$	0,66	0,15	0,5755	0,7408	0,67	0,13	0,6157	0,7308	0,67	0,13	0,6292	0,7069	0,67	0,13	0,6278	0,7001