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The Steiner ratio for points on a triangular lattice*

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Abstract

The study of spanning trees and Steiner trees arises naturally in applications, such as in the design of integrated circuit boards, communication networks, power networks and pipelines of minimum cost. In such applications the Steiner ratio is an indication of how badly a *minimum* spanning tree performs compared to a Steiner *minimal* tree. In this paper a short proof is presented for the Steiner ratio for points on a triangular lattice in the Euclidean plane. A Steiner tree in two dimensions is "lifted" to become a rectilinear tree in three dimensions, where it is altered. The rectilinear tree is then projected back into the plane and the result readily follows. A short note at the end of the paper compares our three-dimensional rectilinear trees to "impossible objects" such as Escher's "Waterfall."

Key words: Spanning tree, Steiner tree, Steiner ratio, rectilinear Steiner tree, hexagonal Steiner tree, equilateral triangular lattice, Escher.

1 Introduction

Let V be a finite, non-empty set of points in the real space $\mathbb{R}^d$. Let an *arc* be a finite union of straight line segments in $\mathbb{R}^d$ which is homeomorphic to the closed unit interval $[0,1]$. Let E be a finite set of arcs such that both endpoints of each arc are elements of V. The set of *vertices* V together with the set of *edges* E are called a *topological graph* (which naturally defines a graph) in general, and a *spanning tree* of V if it is furthermore connected and acyclic. If c is a vector in $\mathbb{R}^d$ and G is a topological graph in $\mathbb{R}^d$, then the topological graph $G+c=\{x+c : x \in G\}$ is called a *translate* of G. A *Steiner tree* of V is a spanning tree of some finite vertex set $V \cup S$ in $\mathbb{R}^d$ where all vertices in S have degree at least 3. The vertices in V are called *terminals* and those in S are called *Steiner points*.

The *length* $\|T\|$ of a tree T is defined as the total length of all its segments. A *minimum spanning tree* (MST) of V is a spanning tree of V of smallest length. A *Steiner minimal tree* (SMT) of V is a Steiner tree of V of smallest length. To see that a SMT exists, note that a Steiner tree with n terminals and m Steiner points has $n+m-1$ edges. Since terminals have degree at least 1 and Steiner points have degree at least 3, there are at least $n/2+3m/2$ edges, and thus $n+m-1 \geq n/2+3m/2$. It follows that there are at most $n-2$

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Steiner points, and thus a finite number of possible graph structures for the Steiner trees of V. A shortest Steiner tree with a specific graph structure will have edges which are straight line segments and Steiner points which are all within a closed ball containing V. If we consider the Steiner points to be variable within such a ball, then the length of the tree is a continuous function defined on a compact set, which has to achieve a minimum.

The *Steiner ratio* ρ is defined as

$$\rho = \sup_{\text{any } V \text{ in } \mathbb{R}^2} \frac{\|\mathrm{MST}(V)\|}{\|\mathrm{SMT}(V)\|},$$

where $\mathrm{MST}(V)$ is an MST of V and $\mathrm{SMT}(V)$ is an SMT of V.

The study of spanning trees and Steiner trees has obvious practical value related to the design of power networks, communication networks and pipelines of minimum cost. It also aids in the design of integrated circuit boards, where shorter networks require less time to charge and discharge, making the circuit boards faster. The Steiner ratio is an indication of how badly a minimum spanning tree will perform compared to a Steiner minimal tree. In practice a spanning tree may indeed sometimes be used instead of a Steiner tree, because a minimum spanning tree can be constructed in polynomial time [9], whereas no such algorithm is known to exist for Steiner minimal trees. (The Euclidean Steiner problem is NP-hard [4].)

In 1968 Gilbert and Pollak [5] conjectured the Steiner ratio to be $2/\sqrt{3}$. The fact that $2/\sqrt{3}$ is a lower bound for the Steiner ratio follows from Figure 1, which shows three equidistant vertices, an SMT with edges meeting at 120°, as well as three dotted lines, any two of which form an MST.

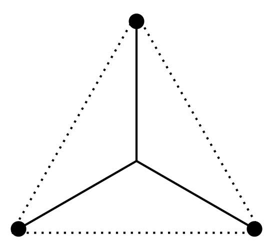

Figure 1: *A Steiner minimal tree.*

It is natural to consider not only three, but also more vertices on an equilateral triangular lattice. It was shown by Du and Hwang [2] that the Steiner ratio for any number of vertices on an equilateral triangular lattice is indeed $2/\sqrt{3}$. The original proof is quite long and incorporates a complicated case analysis. In what follows a shorter proof is presented which is conceptually rather interesting: A Steiner tree in two dimensions is "lifted" to become a rectilinear tree in three dimensions, where it is altered. The rectilinear tree is then projected back into the plane and the result readily follows. The paper closes with a short note which compares three-dimensional rectilinear trees to "impossible objects" such as Escher's "Waterfall."

In 1992 Du and Hwang published a paper [2] and a chapter in a book [3] confirming the correctness of the Gilbert-Pollak conjecture. (The proof for vertices on an equilateral triangular lattice forms an important part of these works.) It has since been shown (see [1] and [7]) that there are fundamental gaps in their argument. The author plans to comment on this extensively in a later paper, but might mention that the general method of Du and Hwang can be adapted for a proof of the Gilbert-Pollak conjecture for 7 points. In this regard their result for vertices on an equilateral triangular lattice has an important consequence.

2 Rectilinear Steiner trees and diagonals

In this section the segments which make up the arcs of a Steiner tree are assumed to be parallel to the x-, y- or z-axis of the space. We refer to such arcs as *rectilinear arcs*. The length of the shortest rectilinear arc between two points is called the *rectilinear distance* between the points. (The norm with which this distance measure is associated is known as the L_1 *norm* or *taxicab norm*.) A Steiner tree consisting only of rectilinear arcs is called a *rectilinear Steiner tree*, and a shortest such tree is called a *rectilinear Steiner minimal tree* (RSMT).

Given n vertices in the plane, a grid can be created by constructing a horizontal line and a vertical line through each vertex. This network is commonly called the *grid graph* of the vertices. The following is a result of Hanan [6], but a new proof is provided.

Lemma 1 *Given n terminals in the plane, then there exists an RSMT with all segments on the grid graph of the terminals. Furthermore, each maximal segment (consisting of a maximal sequence of adjacent collinear segments) contains at least one of the terminals.*

Proof: First consider an RSMT for which the number of horizontal maximal segments that do not contain a terminal is a minimum, and assume it to be greater than zero. Consider the topmost of these maximal segments. Since we have an RSMT, this maximal segment can be moved up or down by a sufficiently small amount Δx without decreasing (or increasing) the length of the tree (Figure 2).

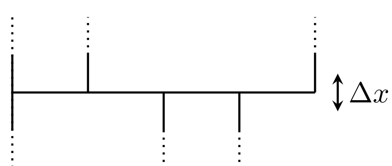

Figure 2: *A maximal segment of an RSMT.*

We move the maximal segment upwards until a terminal or horizontal segment is reached, thus decreasing the number of horizontal maximal segments not containing terminals and providing a contradiction. It follows that there is an RSMT in which each horizontal maximal segment contains a terminal. Among all such RSMTs one may be distinguished in which the number of vertical maximal segments not containing terminals is a minimum. As above, it follows that this number is 0. Hence the RSMT obtained lies on the grid graph. ■

The result of Lemma 1 may be generalized to three dimensions: Given n vertices in $\mathbb{R}^3$ a plane may be constructed perpendicular to each of the axes through each vertex. The intersection of any two planes, with distinct normals, forms a line. The collection of all such lines is known as the *grid graph* of the vertices. By a *maximal planar tree* is meant a tree which lies in a plane perpendicular to one of the axes such that no other tree in the same plane contains it. (See [10] for more on Steiner points in higher dimensions.)

Lemma 2 *Given n terminals in $\mathbb{R}^3$, then there exists a RSMT with all segments on the grid graph of the terminals. Furthermore, each maximal planar tree of the RSMT contains at least one of the terminals.*

Proof: First consider an RSMT for which the number of maximal planar trees in planes perpendicular to the z-axis which do not contain a terminal is a minimum, and assume it to be more than zero. Consider the topmost of these maximal planar trees (*i.e.* with largest z-coordinate). Since we have an RSMT, this maximal planar tree can be moved up or down by a sufficiently small amount Δz without decreasing (or increasing) the length of the tree.

The maximal planar tree may be moved upwards until a terminal or horizontal maximal planar tree is reached, thus decreasing the number of horizontal maximal planar trees not containing terminals and providing a contradiction. It follows that there is an RSMT for which each maximal planar tree which is perpendicular to the z-axis contains a terminal. Among all such RSMTs one may be distinguished in which the number of maximal planar trees perpendicular to the y-axis not containing terminals is a minimum. As above, it follows that this number is 0. Finally the same is done for the x-axis. ■

If a horizontal and a vertical line is constructed through each integer coordinate pair in the plane, an infinite grid graph is obtained. Any translate of this grid graph is called a *square grid*. For three dimensions a *cube grid* is defined similarly, by constructing three lines, parallel to the coordinate axes, through all points with integer coordinates and by considering translates.

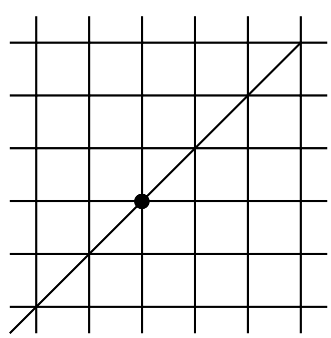

Figure 3: *A square grid.*

In the plane all lines parallel to $y = x$ (*i.e.* parallel to the vector $(1, 1)$) through all integer coordinate pairs, are collectively called *diagonals*. A vertex on one of these diagonals naturally defines a square grid in the plane if an intersection of the square grid coincides with this vertex (see Figure 3). Similarly, *diagonals* in three dimensions pass through all points with integer coordinates and are parallel to the vector $(1,1,1)$. A vertex on a diagonal now naturally defines a cube grid. Note that two points on diagonals in the plane (three dimensional space) define the same square grid (cube grid) if they have the same x or y (x or y or z) coordinate.

Consider the following problem: Given n different diagonals, each with a terminal on it, where should the terminals be for the RSMT to have minimal length? For two dimensions it is not difficult to see that the result of the following lemma is true.

Lemma 3 *Given n terminals on diagonals in the plane, the terminals may be slid along the diagonals to new positions so that they all have the same y-coordinate and so that the new RSMT is not longer than the initial one.* ■

In three dimensions the problem is more complex. Note that the RSMT of $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ has minimal length and that the result of Lemma 3 is not true if the points are moved along diagonals to lie in the same plane. To see why this is so, consider the three diagonals which go through $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Fix one terminal at $(0,0,1)$ while allowing the other two terminals to be moved on their respective diagonals. Next construct around each terminal an octahedron such that all points on the surface of the octahedron are at rectilinear distance 1 from the terminal, as shown in Figure 4.

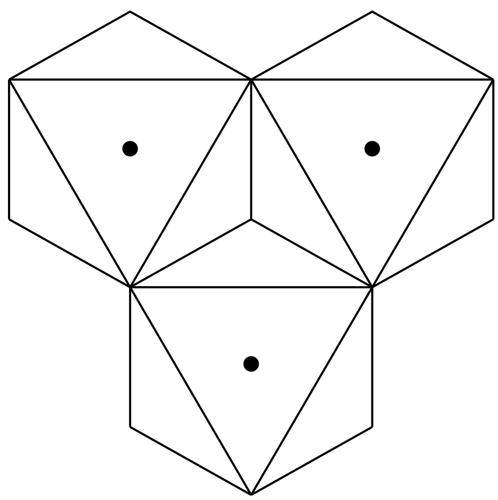

Figure 4: *Octahedrons. Diagonals, as projected onto a plane perpendicular to the vector* $(1,1,1)$, *are indicated by means of dots.*

The only positions for the two terminals not yet fixed which will ensure that the union of the three octahedra is connected, are $(1,0,0)$ and $(0,1,0)$. Now since any RSMT for the three terminals will for each terminal contain a path connecting the terminal to the surface of the octahedron, the RSMT cannot be shorter than 3, and this is only achieved when $(0,0,0)$ is a Steiner point. Finally, the RSMT will at best remain the same if another terminal is introduced, so that the RSMT remains the same after introduction of the terminal $(0,0,0)$.

Lemma 4 *Given n terminals on diagonals in three dimensional space, then we can slide the terminals along the diagonals to new positions so that they all define the same cube grid and so that the new RSMT is not longer than the initial one.*

Proof: Assume that the result of the lemma is false. This implies that if the positions of the terminals on the diagonals are such that the length of the RSMT is minimal, then the terminals define more than one cube grid. Assume that the number of cube grids defined is as small as possible and that the RSMT is in the form described by Lemma 2.

Consider the set A of all terminals which define a particular cube grid. Each horizontal maximal planar tree which contains a terminal or terminals from A may be moved up or down (together with the terminals) by a sufficiently small amount Δz so that the change in the length of the RSMT is linear. If all horizontal maximal planar trees with terminals from A are moved upwards simultaneously, then the change in the length of the RSMT remains linear until some terminal in A has the same z-coordinate as a terminal which is not in A. Let Z be the length of the upward movement for this to happen.

The maximal planar trees which are perpendicular to the x-axis and which contain terminals in A may similarly be moved in a positive direction until some terminal in A has the same x-coordinate as a terminal which is not in A. Let X be the length of this movement and let Y be the length of the corresponding movement in the positive y-direction. The three movements can be combined to achieve movement of the elements of A along the direction of vector $(1, 1, 1)$ with linear change in the length of the RSMT if this Δd is sufficiently small. Since the RSMT has minimal length, the length of the RSMT has to stay constant. Let $D = \min(X, Y, Z)$. If the elements of A are moved by a distance D in the positive x-, y- and z-directions, then an RSMT with the same length is obtained, with all terminals on diagonals, and for which the number of cube grids defined by the terminals is one fewer, providing a contradiction and showing that it is possible for all terminals to define the same cube grid. ■

3 Hexagonal Steiner trees

Given three directions, each two of which form an angle of 120°, a Steiner tree on n points in the plane for which all line segments are parallel to these directions, is called a *hexagonal Steiner tree*. A shortest such tree is called a *hexagonal Steiner minimal tree* (HSMT). A *junction* is either a Steiner point or a non-terminal point where two segments join at different angles. The example in Figure 5 has four junctions. For terminals on an equilateral triangular lattice, the following result holds. (The result is known [2], but a novel proof is provided.)

Lemma 5 *Consider any set of n terminals on an equilateral triangular lattice. Let the three directions for hexagonal Steiner trees be parallel to the edges of the equilateral triangles of the lattice. Then there exists an HSMT for which all junctions are lattice points.*

Proof: The proof proceeds by using Lemma 4. The projection of all diagonals in $\mathbb{R}^3$ onto a plane perpendicular to the vector $(1, 1, 1)$ forms an equilateral triangular lattice.

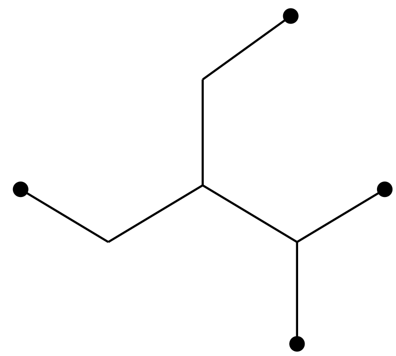

Figure 5: *A hexagonal Steiner tree.*

Furthermore, a hexagonal Steiner tree for n lattice points can be lifted to a rectilinear tree in $\mathbb{R}^3$ with the terminals on diagonals, such that the projection of this rectilinear tree onto the plane parallel to $(1,1,1)$ returns the hexagonal Steiner tree. The desired HSMT is now obtained as follows: Begin with any HSMT, lift this tree into $\mathbb{R}^3$, replace it by a tree of equal length according to Lemma 4 (terminals now all lie on vertices of the same cube grid), modify this tree by using Lemma 2 (all segments now also lie on the cube grid), finally project the tree back to the plane (perpendicular to $(1,1,1)$) and note that this is still an HSMT for which all junctions are now lattice points. ■

4 Vertices on an equilateral triangular lattice

The following lemma is due to Weng [11].

Lemma 6 *Given a set P of vertices in the plane together with the directions for hexagonal Steiner trees, it follows that*

$$\frac{\|HSMT(P)\|}{\|SMT(P)\|} \leq 2/\sqrt{3}.$$

Proof: Note, for a triangle ABC with a $120°$ angle at B, that $\|AB\|+\|BC\| \leq 2/\sqrt{3}\|AC\|$. Now each line of an SMT can be replaced by two lines along the given directions, thus forming a sufficiently short hexagonal Steiner tree for the lemma to hold. ■

A set of vertices on an equilateral triangular lattice is called a *cluster* if the graph obtained by connecting adjacent vertices is connected.

Theorem 1 *For any cluster of vertices on an equilateral triangular lattice the Steiner ratio is $\rho = 2/\sqrt{3}$.*

Proof: Choose the directions for hexagonal Steiner trees parallel to the sides of a smallest triangle in the lattice. From Lemma 5 it follows that all lines of an HSMT connect adjacent vertices on the lattice. It follows that the length of an HSMT is equal to that of an MST. Lemma 6 may now be used to complete the proof. ■

5 Conclusion

It was shown in this paper that it is possible to lift a two dimensional hexagonal tree to obtain a three dimensional rectilinear tree, and that this tree may be altered so that all edges lie on a grid. It is much easier to establish the Steiner ratio for any cluster of vertices on an equivalent triangular lattice adopting this lifting proof technique than adopting a direct case-analysis approach.

It is interesting to note that this lifting technique cannot be applied to a circuit, and that this fact can be used to create an impossible object such as MC Escher's "Waterfall," shown with permission in Figure 6. If any part of the circuit is covered (which turns it into a tree), then the picture can be interpreted as an ordinary object. See [12] for a more detailed discussion of Escher's "Waterfall."

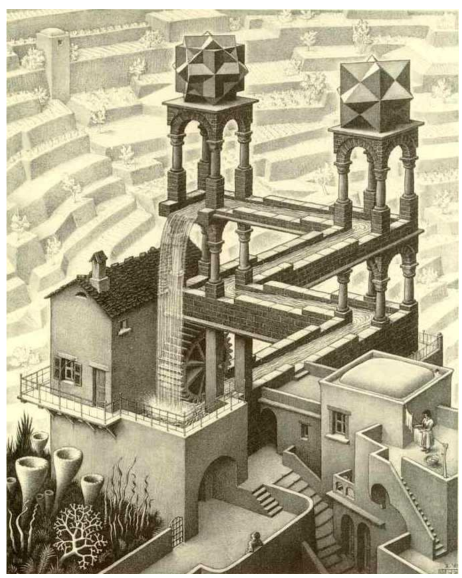

Figure 6: *M.C. Escher's "Waterfall". (©2008 The M.C. Escher Company – the Netherlands [8]. All rights reserved. Used by permission.)*

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