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A retrial queueing-inventory system with J-additional options for service and finite source

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Abstract
A continuous review (s, S) inventory system at a service facility with finite homogeneous sources of demands and retrial is analysed. The lifetime of each item is assumed to be exponential. Before items are delivered to the customers, some basic service on the item must be performed. It is known as a regular or main service. The service may get interrupted according to a Poisson process and it restarts after an exponentially distributed time. If the server is idle at the time of arrival of a customer and the inventory level is positive, then the service begins immediately. After the completion of regular service, a customer may either abandon the system forever or demand for a second service from the same server, which is multi-optional. If any arriving customer finds that the server is busy or inventory level is zero, he/she either enters into the orbit with probability p or balks (does not enter) with probability $1 - p$. The stationary distribution of the number of customers in the system, server status and the inventory level is obtained by the matrix method. The Laplace-Stieltjes transform of the waiting time of the tagged customer is derived. Various system performance measures are derived and the total expected cost rate is computed under a suitable cost structure. A numerical illustration is given.

Key words: (s, S) policy, service interruption, finite source, retrial, repair, essential and optional service.

1 Introduction

In recent years, there has been a considerable interest in the stochastic inventory system in which an item demanded by the customer is not immediately delivered. This situation

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arises when the items in the inventory needs some time for preparation and hence it is considered as having positive service time. Sigman and Simchi-Levi [17] introduced the notion of inventory with positive service time. They assumed that the service time follows an arbitrary distribution and the customers arrive according to a Poisson process wherein the demand is for a single item per customer. Berman *et al.* [3] formulated an inventory model where both demand and service rates are assumed to be deterministic and constant. Later Berman and Kim [4] considered a service facility that provides service to customers using items of inventory with the assumption of Poisson arrival, exponential service times and zero lead times. Berman and Sapna [6] studied a finite capacity inventory model with Poisson arrival, arbitrarily distributed service times and zero lead times. Berman and Kim [5] analysed a queueing-inventory system for service facilities with unlimited waiting space for customers. Maike Schwarz *et al.* [16] have considered a M/M/1 queueing system with attached inventory. They assumed a Poisson demand, exponentially distributed lead times and lost sales for infinite and finite waiting rooms.

In all previously mentioned papers, the researchers analysed a queueing inventory model with a service facility and infinite number of sources. Nevertheless, in many real life situations, it is important to take into account the fact that the number of customers in the source decreases as the number of customers in the system increases. This leads to the study of the inventory model with a finite number of sources. In this paper, a single server queueing-inventory system with a finite number of sources is discussed. A continuous review perishable inventory model with finite population was first initiated by Sivakumar [18]. He assumed that the customers arriving at a stock-out period enter into the orbit and they retry after some random time. The lifetime of each item follows an exponential distribution. Following this, a number of papers on inventory models with a finite population have emerged. Multi-server retrial inventory system with a finite number of sources was studied by Yadavalli *et al.* [24] in which the customers arrive according to a quasi random process. The service times and the lead times were assumed to be exponential. Shophia Lawrence *et al.* [15] analysed a service facility with a finite population in which items in inventory are perishable and customer demand is satisfied only after performing some service on it. The service time and the lead time have a Phase type distribution. The lifetime of each item follows a negative exponential distribution.

Many researchers have studied retrial queueing systems with a finite and infinite number of sources extensively. Artalejo [1], Falin & Artalejo [7] and Falin & Templeton [8] provide reviews on this queueing system. A numerical illustration of inventory systems with retrial was studied by Artalejo *et al.* [2]. The study of inventory models with server interruptions is a topic that has received considerable attention in the last decade. The inventory model with instantaneous replenishment is discussed by Krishnamoorthy *et al.* [12] wherein the service is subject to interruptions. Krishnamoorthy *et al.* [13] made an extensive study of an (s, S) inventory model with the assumption of Poisson arrival and exponential service time. According to a Poisson process the service may get interrupted and it restarts after an exponentially distributed time. Yadavalli *et al.* [22] analysed a finite source perishable inventory system with a service facility having two heterogeneous servers and repeated attempts. They assumed that the first server is perfectly reliable and the second server is subject to interruptions.

From the above papers, the customer will move away from the system after completion of regular or essential service. However, in day-to-day life, it does not always happen. In some queueing situations all arriving customers require the essential service, whereas a few of them may further demand the subsidiary service provided by the same server immediately after completion of the regular service. Several researchers have studied the concept of additional optional service with queues [11, 19, 20]. Jeganathan [9] investigated a continuous review perishable (s, S) inventory system with N optional services, in which some of the arriving customers asked for second optional service as soon as the completion of first essential service and the second service is multi-optional. He assumed that the customer arrivals follow a Poisson process. Yadavalli & Jeganathan [23] analysed a finite source perishable inventory system with second optional service and server interruptions. Recently, Jeganathan *et al.* [10] analysed a retrial queueing-inventory system with priority customers and second optional service.

A finite population Markovian inventory system with server interruptions, multi-optional service and repeated attempts is considered in this paper. The rest of the paper is organized as follows. A detailed description of the model is explained in §2. In §3, the mathematical solution of the model is carried out. The Laplace-Stieltjes transform of waiting time distribution for customers in the orbit is derived in §4. Some key system performance measures are obtained in §5, while §6 is dedicated to cost analysis and sensitivity investigation. Conclusions are given in the final section.

2 Model description

In this investigation, a finite source Markovian inventory system is considered with the following assumptions. Consider a single server perishable inventory system, where the primary arrivals are generated from N, $1 < N < \infty$ homogeneous sources. The inventory is replenished according to (s, S) ordering policy. According to this, an order for $Q(= S - s > s + 1)$ items are placed when the on-hand inventory level falls below s. The requirement $S - s > s + 1$ assures that after a replenishment, the inventory level will be greater than the reorder level. Otherwise it is impossible to place a reorder which leads to perpetual shortage. The positive lead-time of the replenishment is assumed to be negative exponential with the rate $\beta(> 0)$. The lifetime of each commodity has a negative exponential distribution with the parameter $\gamma(> 0)$. It is assumed that during service, the items in inventory cannot be perished.

The server can retain in three states, namely idle, busy and interruption. Likewise, each source can also be in three states, namely free, retrial and under service.

1. If a source is in free state at time t a primary customer can arrive during the interval $(t, t + dt)$ with probability $\lambda dt + o(t)$. It is assumed that the waiting space is not available in front of the server. When the server is in idle state and the inventory level is positive then the demand is served immediately. Hence, the source moves into "under service" state and the server moves into "busy" state.
2. If the demand finds the inventory level zero or server busy or server is on interruption at the moment of their arrival, then they either abandon (with non-zero probability

p) the area of service and join a pool of blocked customers called orbit or balk the system (with probability $1-p$).

In this paper, the classical retrial method is followed. More explicitly, the probability of a repeated attempt in an interval $(t, t+dt)$ given that i customers are in the orbit attempt is $i\theta + o(dt)$. If the server is idle with positive inventory then it provides the first essential service (FES) to all arriving customers. The first essential service is also referred to as regular service or main service. It is assumed to be exponentially distributed with parameter μ_0. As soon as the first essential service of each demand is completed, then with probability $r_k, 1 \leq k \leq J$, a customer may opt to receive a second optional service (SOS) from $J(J \geq 1)$ kinds of different services (*i.e.*, Type1, Type2,..., Type J), or else, with probability r_0=$1-\sum_{k=1}^{J} r_k$ the customer may opt to abandon the system and then the server becomes idle. The service time of the second optional service is assumed to be distributed exponentially with parameter $\mu_k, k = 1, 2, \ldots, J$.

When the server provides the first essential service to a customer, the service may get interrupted according to a Poisson process of rate α_0. In the second multi-optional service, the service may get interrupted with an exponential rate $\alpha_k, 1 \leq k \leq J$. It is assumed that while the server is under interruption, no further interruption can befall the server. The repair time of both service phases (first essential service and second multi- optional service) are assumed to be exponentially distributed with parameters η_0 and $\eta_k, 1 \leq k \leq J$, respectively. Figure 1 shows a typical picture of the model. All random variables are independent of each other.

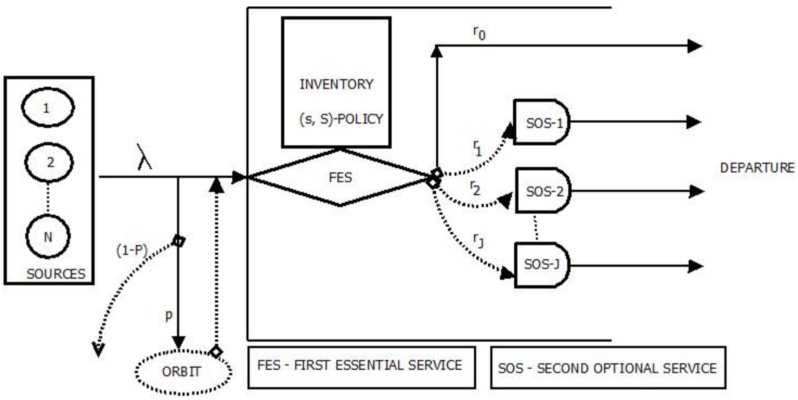

Figure 1: Dynamics of the queueing inventory system with finite source.

3 Mathematical formulation of the model

Let $l(t)$ and $x(t)$ indicate the inventory level and the number of customers in the orbit at time t, $l(t) \in \{0, 1, 2, \ldots, S\}$ and $x(t) \in \{0, 1, 2, \ldots, N\}$, and let $y(t)$ denote the status of the server by

$$y(t) = \begin{cases} S_0, & \text{if the server is idle at time } t, \\ S_1, & \text{if the server is busy with FES at time } t, \\ S_2, & \text{if the server is on interruption during FES at time } t, \\ 1, & \text{if the server is busy with Type 1 service at time } t, \\ 2, & \text{if the server is busy with Type 2 service at time } t, \\ \vdots & \vdots \\ J-1, & \text{if the server is busy with Type } J-1 \text{ service at time } t, \\ J, & \text{if the server is busy with Type } J \text{ service at time } t, \\ J+1, & \text{if the server is on interruption during Type 1 service at time } t, \\ J+2, & \text{if the server is on interruption during Type 2 service at time } t, \\ \vdots & \vdots \\ 2J-1, & \text{if the server is on interruption during Type } J-1 \text{ service at time } t, \\ 2J, & \text{if the server is on interruption during Type } J \text{ service at time } t. \end{cases}$$

The activities of the system can be expressed by a three-dimensional stochastic process $\{a(t) = (l(t), y(t), x(t)), t \geq 0\}$, with the finite discrete state space E, where

$$\begin{aligned} E &= \{0 \leq i_1 \leq S,\ i_2 = S_0,\ 0 \leq i_3 \leq N\} \cup \\ &\quad \{1 \leq i_1 \leq S,\ i_2 = S_1, S_2,\ 0 \leq i_3 \leq N-1\} \cup \\ &\quad \{0 \leq i_1 \leq S,\ 1 \leq i_2 \leq 2J,\ 0 \leq i_3 \leq N-1\}. \end{aligned}$$

Due to the assumptions of a Poisson arrival process, exponentially distributed service times, the replenishment process, the repair process and the interruption process $\{a(t), t \geq 0\}$ is a Homogeneous Continuous Time Markov Chain (HCTMC). Its limiting distribution is indicated by $\pi(i_1, i_2, i_3)$:

$$\pi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr[l(t) = i_1, y(t) = i_2, x(t) = i_3 | l(0), y(0), x(0)].$$

In the sequel, I_k refers to an identity matrix of order k, $\mathbf{e}$ refers to a column vector of appropriate dimension containing all ones, $[A]_{ij}$ denotes the entry at $(i,j)^{th}$ position of a matrix A, δ is the delta function defined by $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$, $H(x)$ is the Heaviside function *i.e.* $H(x) = 1$ if $x \geq 0$, otherwise $H(x) = 0$ and $k \in V_i^j$ denotes $k = i, i+1, \ldots j$.

The steady-state equation of $\{a(t) = (l(t), y(t), x(t)), t \geq 0\}$ satisfies the following balance equations.

For $i_1 \in V_0^S$, $\quad i_2 = S_0$, and $i_3 \in V_0^N$,

$$\begin{aligned} ((N-i_3)\lambda + H(s-i_1)\beta + i_1\gamma + \bar{\delta}_{i_1 0} i_3\theta)\pi^{(i_1,i_2,i_3)} &= \bar{\delta}_{i_1 S}(i_1+1)\gamma\pi^{(i_1+1,i_2,i_3)} + \\ \bar{\delta}_{i_3 0}\delta_{i_1 1} p(N-i_3)\lambda\pi^{(i_1-1,i_2,i_3-1)} &+ H(i_1-Q)\beta\pi^{(i_1-S+s,i_2,i_3)} + \\ \bar{\delta}_{i_3 N}\bar{\delta}_{i_1 S} r_0\mu_0\pi^{(i_1+1,S_1,i_3)} &+ \bar{\delta}_{i_3 N}\sum_{k=1}^{J}\mu_k\pi^{(i_1,k,i_3)}. \end{aligned} \tag{1}$$

For $i_1 \in V_1^S, \quad i_2 = S_1$, and $i_3 \in V_0^{N-1}$,

$$\begin{aligned}(p(N-(i_3+1))\lambda + H(s-i_1)\beta + \bar{\delta}_{i_1 1}(i_1-1)\gamma + \sum_{k=0}^{J} r_k\mu_0 + \alpha_0)\pi^{(i_1,i_2,i_3)} = \\ \bar{\delta}_{i_3 0}p(N-i_3)\lambda\pi^{(i_1,i_2,i_3-1)} + H(i_1-(Q+1))\beta\pi^{(i_1-S+s,i_2,i_3)} + \\ \bar{\delta}_{i_1 S}i_1\gamma\pi^{(i_1+1,i_2,i_3)} + (i_3+1)\theta\pi^{(i_1+1,S_0,i_3+1)} + (N-i_3)\lambda\pi^{(i_1,S_0,i_3)} + \eta_0\pi^{(i_1,S_2,i_3)}.\end{aligned} \tag{2}$$

For $i_1 \in V_1^S, \quad i_2 = S_2$, and $i_3 \in V_0^{N-1}$,

$$\begin{aligned}(p(N-(i_3+1))\lambda + H(s-i_1)\beta + \bar{\delta}_{i_1 1}(i_1-1)\gamma + \eta_0)\pi^{(i_1,i_2,i_3)} = \bar{\delta}_{i_3 0}p(N-i_3)\lambda\pi^{(i_1,i_2,i_3-1)} \\ +H(i_1-(Q+1))\beta\pi^{(i_1-S+s,i_2,i_3)} + \bar{\delta}_{i_1 S}i_1\gamma\pi^{(i_1+1,i_2,i_3)} + \alpha_0\pi^{(i_1,S_1,i_3)}.\end{aligned} \tag{3}$$

For $i_1 \in V_0^S, \quad i_2 \in V_1^J$, and $i_3 \in V_0^{N-1}$,

$$\begin{aligned}(p(N-(i_3+1))\lambda + H(s-i_1)\beta + \bar{\delta}_{i_1 0}i_1\gamma + \sum_{k=1}^{J}(\delta_{i_2 k}\mu_k) + \sum_{k=1}^{J}(\delta_{i_2 k}\alpha_k))\pi^{(i_1,i_2,i_3)} = \\ \bar{\delta}_{i_3 0}p(N-i_3)\lambda\pi^{(i_1,i_2,i_3-1)} + H(i_1-Q)\beta\pi^{(i_1-S+s,i_2,i_3)} + \bar{\delta}_{i_1 S}(i_1+1)\gamma\pi^{(i_1+1,i_2,i_3)} \\ +\bar{\delta}_{i_1 0}\{\sum_{k=1}^{J} r_k\mu_0\}\pi^{(i_1,S_1,i_3)} + \{\sum_{k=1}^{J}\delta_{i_2 J+k}\eta_k\}\pi^{(i_1,J+k,i_3)}.\end{aligned} \tag{4}$$

For $i_1 \in V_0^S, \quad i_2 \in V_{J+1}^{2J}$, and $i_3 \in V_0^{N-1}$,

$$\begin{aligned}(p(N-(i_3+1))\lambda + H(s-i_1)\beta + \bar{\delta}_{i_1 0}i_1\gamma + \sum_{k=1}^{J}(\delta_{i_2 k}\eta_k))\pi^{(i_1,i_2,i_3)} = \\ \bar{\delta}_{i_3 0}p(N-i_3)\lambda\pi^{(i_1,i_2,i_3-1)} + H(i_1-Q)\beta\pi^{(i_1-S+s,i_2,i_3)} + \bar{\delta}_{i_1 S}(i_1+1)\gamma\pi^{(i_1+1,i_2,i_3)} \\ +\{\sum_{k=J+1}^{2J}\delta_{i_2 k}\alpha_{(i_2-J)}\}\pi^{(i_1,i_2-J,i_3)}.\end{aligned} \tag{5}$$

To mark the steady-state equations in expressions of matrix form, states in E are rescheduled in the following lexicographical ordering:

$$\lll i_1 \ggg \; = \; \begin{cases} \ll i_1, i_2, \gg, & i_1 = 1,2,\dots S,\; i_2 = S_0, S_1, S_2, 1, 2, \dots, 2J; \\ \ll i_1, i_2, \gg, & i_1 = 0,\; i_2 = S_0, 1, 2, \dots, 2J; \end{cases}$$

$$\ll i_1, i_2 \gg \; = \; \begin{cases} < i_1, S_0, 0 >, < i_1, S_0, 1 >, \dots, < i_1, S_0, N >, & i_1 = 0,1,2,\dots S; \\ < i_1, i_2, 0 >, < i_1, i_2, 1 >, \dots, < i_1, i_2, N-1 >, & i_1 = 1,2,\dots S, \\ & i_2 = S_1, S_2; \\ < i_1, i_2, 0 >, < i_1, i_2, 1 >, \dots, < i_1, i_2, N-1 >, & i_1 = 0,1,2,\dots S, \\ & i_2 = 1,2,\dots,2J; \end{cases}$$

and define the corresponding vectors

$$\begin{aligned}
\mathbf{\Pi^{(0)}} &= (\mathbf{\Pi^{(0,S_0)}}, \mathbf{\Pi^{(0,1)}}, \mathbf{\Pi^{(0,2)}}, \dots, \mathbf{\Pi^{(0,2J)}}), \\
\mathbf{\Pi^{(i_1)}} &= (\mathbf{\Pi^{(i_1,S_0)}}, \mathbf{\Pi^{(i_1,S_1)}}, \mathbf{\Pi^{(i_1,S_2)}}, \mathbf{\Pi^{(i_1,1)}}, \dots, \mathbf{\Pi^{(i_1,2J)}}), \quad i_1 \in V_1^S; \\
\mathbf{\Pi^{(0,S_0)}} &= (\pi^{(0,S_0,0)}, \pi^{(0,S_0,1)}, \dots, \pi^{(0,S_0,N)}), \\
\mathbf{\Pi^{(0,i_2)}} &= (\pi^{(0,i_2,0)}, \pi^{(0,i_2,1)}, \dots, \pi^{(0,i_2,N-1)}), \ i_2 \in V_1^{2J}; \\
\mathbf{\Pi^{(i_1,S_0)}} &= (\pi^{(i_1,S_0,0)}, \pi^{(i_1,S_0,1)}, \dots, \pi^{(i_1,S_0,N)}), \ i_1 \in V_1^S; \\
\mathbf{\Pi^{(i_1,S_1)}} &= (\pi^{(i_1,S_1,0)}, \pi^{(i_1,S_1,1)}, \dots, \pi^{(i_1,S_1,N-1)}), i_1 \in V_1^S; \\
\mathbf{\Pi^{(i_1,S_2)}} &= (\pi^{(i_1,S_2,0)}, \pi^{(i_1,S_2,1)}, \dots, \pi^{(i_1,S_2,N-1)}), i_1 \in V_1^S; \\
\mathbf{\Pi^{(i_1,i_2)}} &= (\pi^{(i_1,i_2,0)}, \pi^{(i_1,i_2,1)}, \dots, \pi^{(i_1,i_2,N-1)}), \ i_1 \in V_1^S, \ i_2 \in V_1^{2J}.
\end{aligned}$$

Then, $\mathbf{\Pi^{(i_1)}}$ is the probability vector i_1th inventory level with each one element, stating a particular mixture of the inventory level and status of the server and the number of customers in the orbit. Using the vectors $\mathbf{\Pi^{(i_1)}}, i_1 \in V_0^S$, the system of linear equations (3.1) – (3.5) can be written as follows:

$$\begin{aligned}
\mathbf{\Pi^{i_1}} D_{i_1} + \mathbf{\Pi^{i_1-1}} C_{i_1-1} &= \mathbf{0}, \qquad i_1 = 1, 2, \dots, Q, \\
\mathbf{\Pi^{i_1}} D_{i_1} + \mathbf{\Pi^{i_1-1}} C_{i_1-1} + \mathbf{\Pi^{(i_1-1-Q)}} A_1 &= \mathbf{0}, \qquad i_1 = Q+1, \\
\mathbf{\Pi^{i_1}} D_{i_1} + \mathbf{\Pi^{i_1-1}} C_{i_1-1} + \mathbf{\Pi^{(i_1-1-Q)}} A &= \mathbf{0}, \qquad i_1 = Q+2, Q+3, \dots, S, \\
\mathbf{\Pi^{S}} C_S + \mathbf{\Pi^{s}} A &= \mathbf{0}.
\end{aligned}$$

Then, the limiting distribution vector $\mathbf{\Pi} = (\mathbf{\Pi^{(0)}}, \mathbf{\Pi^{(1)}}, \dots, \mathbf{\Pi^{(S)}})$ is the unique key of the system

$$\mathbf{\Pi}\Theta = \mathbf{0} \tag{6}$$

and the normalization condition

$$\mathbf{\Pi}\boldsymbol{e} = \sum_{(i_1,i_2,i_3)}\sum\sum \pi^{(i_1,i_2,i_3)} = 1, \tag{7}$$

where $\Theta = ((d((i_1,i_2,i_3),(j_1,j_2,j_3)))),\ (i_1,i_2,i_3),(j_1,j_2,j_3) \in E$ is the infinitesimal generator matrix of $\{a(t), t \geq 0\}$ can be expressed as follows:

$$\Theta = \begin{array}{c} \lll S \ggg \\ \lll S-1 \ggg \\ \vdots \\ \lll s+1 \ggg \\ \lll s \ggg \\ \lll s-1 \ggg \\ \vdots \\ \lll 1 \ggg \\ \lll 0 \ggg \end{array}
\begin{pmatrix}
C_S & D_S & & & & & & & & \\
 & C_{S-1} & D_{S-1} & & & & & & & \\
 & & \cdots & & & & & & & \\
 & & & \cdots & C_{s+1} & D_{s+1} & & & & \\
A & & & & & C_s & D_s & & & \\
 & A & & & & & C_{s-1} & & & \\
 & & \cdots & & & & \cdots & & & \\
 & & A & & & & & \cdots & C_1 & D_1 \\
 & & & A_1 & & & & & & C_0
\end{pmatrix}$$

where $A_1 = [a_{1(i_1,i_2,i_3),(j_1,j_2,j_3)}]$ is a rectangular matrix of size $(2J+1)N+1 \times (2J+3)N+1$ and its elements are given by

$$a_{1(i_1,i_2,i_3),(j_1,j_2,j_3)} = \begin{cases} \beta, & j_1 = Q, \quad j_2 = i_2, \quad j_3 = i_3, \\ & i_1 = 0, \quad i_2 = S_0, \quad i_3 \in V_0^N, \\ & \\ & j_1 = Q, \quad j_2 = i_2 \quad j_3 = i_3, \\ & i_1 = 0, \quad i_2 \in V_1^{2J}, \quad i_4 \in V_0^{N-1}, \\ & \\ 0, & \text{otherwise.} \end{cases}$$

The matrix A is a square matrix of size $((2J+3)N+1)$ and it can be expressed as $A = \beta \times I_{((2J+3)N+1)}$.

The matrices $D_{i_1} = [d^{i_1}{}_{(i_1,i_2,i_3),(j_1,j_2,j_3)}], i_1 \in V_1^S$ govern the transitions from the state $\lll i_1 \ggg$ into the state $\lll i_1 - 1 \ggg$ and, hence D_1 is a rectangular matrix of size $(2J+3)N+1 \times (2J+1)N+1$ and other matrices $D_{i_1}, i_1 \in V_2^S$ are square matrices of size $((2J+3)N+1)$ with their the following elements, for $k \in V_1^J$

$$d^{i_1}{}_{(i_1,i_2,i_3),(j_1,j_2,j_3)} = \begin{cases} i_1\gamma, & j_1 = i_1 - 1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 = S_0, \quad i_3 \in V_0^N, \\ & \\ & j_1 = i_1 - 1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 \in V_1^{2J}, \quad i_3 \in V_0^{N-1}, \\ & \\ (i_1 - 1)\gamma, & j_1 = i_1 - 1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & i_1 \in V_2^S, \quad i_2 = S_1, S_2 \quad i_3 \in V_0^{N-1}, \\ r_0\mu_0, & j_1 = i_1 - 1, \quad j_2 = S_0, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 = S_1, \quad i_3 \in V_0^{N-1}, \\ & \\ r_k\mu_0, & j_1 = i_1 - 1, \quad j_2 = k, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 = S_1, \quad i_3 \in V_0^{N-1}, \\ & \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the matrices $C_{i_1} = [c^{i_1}{}_{(i_1,i_2,i_3),(j_1,j_2,j_3)}], i_1 \in V_0^S$ represents all transitions within $\lll i_1 \ggg$ and C_0 is a square matrix of order $(N(2J+1))+1$ and $C_{i_1}, i_1 \in V_1^S$ are square matrices of order $(N(2J+3))+1$ with the following elements, for $i_1 = 0, \ k \in V_1^{2J}$

$$c^{0}{}_{(i_1,i_2,i_3),(j_1,j_2,j_3)} = \begin{cases} p(N - i_3)\lambda, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3 + 1, \\ & i_1 = 0, \quad i_2 = S_0, \quad i_3 \in V_0^{N-1}, \\ & \\ p(N - (i_3 + 1))\lambda, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3 + 1, \\ & i_1 = 0, \quad i_2 \in V_1^{2J}, \quad i_3 \in V_0^{N-2}, \end{cases}$$

$$\begin{cases} \mu_{i_2}, & j_1 = i_1, \quad j_2 = S_0, \quad j_3 = i_3, \\ & i_1 = 0, \quad i_2 \in V_1^J \quad i_3 \in V_0^{N-1}, \\ \alpha_{i_2}, & j_1 = i_1, \quad j_2 = i_2 + J, \quad j_3 = i_3, \\ & i_1 = 0, \quad i_2 \in V_1^J, \quad i_3 \in V_0^{N-1}, \\ \eta_{i_2}, & j_1 = i_1, \quad j_2 = i_2 - J, \quad j_3 = i_3, \\ & i_1 = 0, \quad i_2 \in V_{J+1}^{2J}, \quad i_3 \in V_0^{N-1}, \\ -h_1, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & i_1 = 0, \quad i_2 = S_0, \quad i_3 \in V_0^N, \\ -h_2, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & i_1 = 0, \quad i_2 \in V_1^J, \quad i_3 \in V_0^{N-1}, \\ -h_3, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & i_1 = 0, \quad i_2 \in V_{J+1}^{2J}, \quad i_3 \in V_0^{N-1}, \\ 0, & \text{otherwise.} \end{cases}$$

where $h_1 = (p(N - i_3)\lambda\bar{\delta}_{i_3 N} + \beta), h_2 = (p(N - (i_3 + 1))\lambda\bar{\delta}_{i_3(N-1)} + \beta + (\alpha_k + \mu_k)\delta_{i_2 k}), h_3 = (p(N - (i_3 + 1))\lambda\bar{\delta}_{i_3(N-1)} + \beta + \eta_(k - J)\delta_{i_2 k})$.

For $i_1 \in V_1^S,\ k \in V_1^{2J}$

$$c^{i_1}{}_{(i_1,i_2,i_3),(j_1,j_2,j_3)} = \begin{cases} (N - i_3)\lambda, & j_1 = i_1, \quad j_2 = S_1, \quad j_3 = i_3 + 1, \\ & i_1 \in V_1^S, \quad i_2 = S_0, \quad i_3 \in V_0^{N-1}, \\ p(N - (i_3 + 1))\lambda, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3 + 1, \\ & i_1 \in V_1^S, \quad i_2 = S_1, S_2, i_2 \in V_1^{2J}, \quad i_3 \in V_0^{N-2}, \\ i_3\theta, & j_1 = i_1, \quad j_2 = S_1, \quad j_3 = i_3 - 1, \\ & i_1 \in V_1^S, \quad i_2 = S_0, \quad i_3 \in V_1^N, \\ \alpha_0, & j_1 = i_1, \quad j_2 = S_2, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 = S_1 \quad i_3 \in V_0^{N-1}, \\ \eta_0, & j_1 = i_1, \quad j_2 = S_1, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 = S_2, \quad i_3 \in V_0^{N-1}, \\ \mu_{i_2}, & j_1 = i_1, \quad j_2 = S_0, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 \in V_1^J, \quad i_3 \in V_0^{N-1}, \end{cases}$$

$$\begin{cases} \alpha_{i_2}, & j_1 = i_1, \quad j_2 = i_2 + J, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 \in V_1^J, \quad i_3 \in V_0^{N-1}, \\ \eta_{i_2}, & j_1 = i_1, \quad j_2 = i_2 - J, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 \in V_{J+1}^{2J}, \quad i_3 \in V_0^{N-1}, \\ -k_1, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & i_1 \in V_1^S, \quad i_2 = S_0, \quad i_3 \in V_0^N, \\ -k_2, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & \in V_1^S, \quad i_2 = S_1, S_2, \quad i_3 \in V_0^{N-1}, \\ -k_3, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & \in V_1^S, \quad i_2 \in V_1^J, \quad i_3 \in V_0^{N-1}, \\ -k_4, & j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \\ & \in V_1^S, \quad i_2 \in V_{J+1}^{2J}, \quad i_3 \in V_0^{N-1}, \\ 0, & \text{otherwise.} \end{cases}$$

where $k_1 = ((N - i_3)\lambda\bar{\delta}_{i_3 N} + i_1\gamma + H(s - i_1)\beta + i_3\theta), k_2 = (p(N - (i_3 + 1))\lambda\bar{\delta}_{i_3(N-1)} + H(s - i_1)\beta + (i_1 - 1)\gamma + (\alpha_0 + \mu_0)\delta_{i_2 S_1} + \delta_{i_2 S_2}\eta_0), k_3 = (p(N - (i_3 + 1))\lambda\bar{\delta}_{i_3(N-1)} + H(s - i_1)\beta + i_1\gamma + (\alpha_k + \mu_k)\delta_{i_2 k}), k_4 = (p(N - (i_3 + 1))\lambda\bar{\delta}_{i_3(N-1)} + H(s - i_1)\beta + i_1\gamma + \eta_{(k-J)}\delta_{i_2 k})$.

A recursive algorithm is now derived for the solutions of the steady-state equations (3.6) and (3.7). The steady state probability vector $\mathbf{\Pi}^{(i_1)}$, $i_1 \in V_0^S$ can be determined from an algorithm given by the following steps.

Step 1. To obtain the value of $\mathbf{\Pi}^{\boldsymbol{Q}}$, the following system of equations is described:

$$\mathbf{\Pi}^{\boldsymbol{Q}} \left[\left\{ (-1)^Q \sum_{j=0}^{s-1} \left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} D_k C_{k-1}^{-1} \right) A C_{S-j}^{-1} \left(\mathop{\Omega}_{l=S-j}^{Q+2} D_l C_{l-1}^{-1} \right) \right] \right\} D_{Q+1} + C_Q + \left\{ (-1)^Q \mathop{\Omega}_{j=Q}^{1} D_j C_{j-1}^{-1} \right\} A \right] = \mathbf{0},$$

and

$$\mathbf{\Pi}^{\boldsymbol{Q}} \left[\sum_{i_1=0}^{Q-1} \left((-1)^{Q-i_1} \mathop{\Omega}_{j=Q}^{i_1+1} D_j C_{j-1}^{-1} \right) + I + \sum_{i_1=Q+1}^{S} \left((-1)^{2Q-i_1+1} \sum_{j=0}^{S-i_1} \left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} D_k C_{k-1}^{-1} \right) A C_{S-j}^{-1} \left(\mathop{\Omega}_{l=S-j}^{i_1+1} D_l C_{l-1}^{-1} \right) \right] \right) \right] \boldsymbol{e} = 1.$$

Step 2. Next, the following values are computed by

$$\begin{aligned}
\mathbf{\Omega_{i_1}} &= (-1)^{Q-i_1}\mathbf{\Pi^{Q}} \underset{j=Q}{\overset{i_1+1}{\Omega}} D_j C_{j-1}^{-1}, \qquad\qquad i_1 = Q-1, Q-2, \ldots, 0 \\
&= (-1)^{2Q-i_1+1}\mathbf{\Pi^{Q}} \sum_{j=0}^{S-i_1} \left[\left(\underset{k=Q}{\overset{s+1-j}{\Omega}} D_k C_{k-1}^{-1} \right) A C_{S-j}^{-1} \left(\underset{l=S-j}{\overset{i_1+1}{\Omega}} D_l C_{l-1}^{-1} \right) \right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad i_1 = S, S-1, \ldots, Q+1 \\
&= I, \qquad\qquad\qquad\qquad\qquad\qquad\quad i_1 = Q.
\end{aligned}$$

Step 3. By $\mathbf{\Pi^{(Q)}}$ and $\Omega_{i_1}, i_1 = 0, 1, \ldots, S$, get the value of $\mathbf{\Pi^{(i_1)}}, i_1 = 0, 1, \ldots, S$. Explicitly,

$$\mathbf{\Pi^{(i_1)}} = \mathbf{\Pi^{(Q)}}\Omega_{i_1}, \quad i_1 = 0, 1, \ldots, S.$$

4 Waiting time analysis of an orbital customer

In this section, the waiting time distribution of an orbital customer is discussed that is specified as the time between the arrival times of the customer and moment upon which he/she gets service. This continuous time random variable is symbolized as W. The objective is to calculate the probability distribution of W and to calculate n^{th} order moments of W. If the arriving customer finds the system in state $(i_1, S_0, i_3),\ i_1 \in V_1^S,\ i_3 \in V_0^{N-1}$, he/she gets the service immediately. Therefore, $W = 0$. The probability for the customer does not wait is denoted by $P\{W = 0\}$ and is given by,

$$P\{W = 0\} = \sum_{i_1=1}^{S} \sum_{i_3=0}^{N-1} \pi^{(i_1, S_0, 0)}.$$

To obtain the distribution of W, some auxiliary variables are defined. Assume that the system is in the state $(i_1, i_2, i_3), i_3 > 0$ at an arbitrary time t. Let $W^*(y) = E[e^{-yW}]$ be the Laplace-Stieltjes transform of the unconditional waiting time and let $W^*_{(i_1,i_2,i_3)}(y) = E[e^{-yW_{(i_1,i_2,i_3)}}]$ be the Laplace-Stieltjes transform of the conditional waiting time. Thus,

$$\begin{aligned}
W^*(y) = &\sum_{i_1=1}^{S} \sum_{i_3=0}^{N-1} \pi^{(i_1,S_0,i_3)} + \sum_{i_3=0}^{N-1} \pi^{(0,S_0,i_3)} W^*_{(0,S_0,i_3+1)}(y) + \sum_{i_1=1}^{S} \sum_{i_3=0}^{N-2} \pi^{(i_1,S_1,i_3)} W^*_{(i_1,S_1,i_3+1)}(y) \\
&+ \sum_{i_1=1}^{S} \sum_{i_3=0}^{N-2} \pi^{(i_1,S_2,i_3)} W^*_{(i_1,S_2,i_3+1)}(y) + \sum_{i_1=0}^{S} \sum_{i_2=1}^{2J} \sum_{i_3=0}^{N-2} \pi^{(i_1,i_2,i_3)} W^*_{(i_1,i_2,i_3+1)}(y).
\end{aligned} \tag{8}$$

To derive $W^*_{(i_1,i_2,i_3)}$, an auxiliary Markov chain is introduced on the state space

$$\begin{aligned}
E^* = &\{*\} \cup \{0,\ i_2 = S_0,\ 0 \le i_3 \le N\} \cup \\
&\{1 \le i_1 \le S,\ i_2 = S_1, S_2,\ 0 \le i_3 \le N-1\} \cup \\
&\{0 \le i_1 \le S,\ 1 \le i_2 \le 2J,\ 0 \le i_3 \le N-1\},
\end{aligned}$$

where $\{*\}$ represents an absorbing state. The chain is on a state (i_1, i_2, i_3), a first-step argument is applied in the auxiliary chain to resolve $W^*_{(i_1,i_2,i_3)}(y)$. Then (see [14], Theorem 6.21) the functions $W^*_{(i_1,i_2,i_3)}(y)$, $(i_1, i_2, i_3) \in E^*$ are the least non-negative solution to the system.

For $0 \le i_1 \le S, \quad i_2 = S_0, \quad 1 \le i_3 \le N,$

$$\begin{aligned} w_1 W^*_{(i_1,i_2,i_3)}(y) &= p(N-i_3)\lambda\bar{\delta}_{i_3 N}\delta_{i_1 0} W^*_{(i_1,S_0,i_3+1)}(y) + (N-i_3)\lambda\bar{\delta}_{i_3 N}\bar{\delta}_{i_1 0} W^*_{(i_1,S_1,i_3)}(y) \\ &+ i_1\gamma\bar{\delta}_{i_1 0} W^*_{(i_1-1,S_0,i_3)}(y) + \beta H(s-i_1) W^*_{(i_1+Q,S_0,i_3)}(y) + i_3\theta\bar{\delta}_{i_1 0} W^*_{(i_1,S_1,i_3-1)}(y) \end{aligned} \tag{9}$$

where

$$w_1 = y + p(N-i_3)\lambda\bar{\delta}_{i_3 N}\delta_{i_1 0} + (N-i_3)\lambda\bar{\delta}_{i_3 N}\bar{\delta}_{i_1 0} + i_1\gamma\bar{\delta}_{i_1 0} + \beta H(s-i_1) + i_3\theta\bar{\delta}_{i_1 0}.$$

For $1 \le i_1 \le S, \quad i_2 = S_1, S_2 \quad 1 \le i_3 \le N-1$

$$\begin{aligned} w_2 W^*_{(i_1,i_2,i_3)}(y) &= p(N-(i_3+1))\lambda\bar{\delta}_{i_3 (N-1)} W^*_{(i_1,i_2,i_3+1)}(y) + \bar{\delta}_{i_1 1}(i_1-1)\gamma W^*_{(i_1-1,i_2,i_3)}(y) \\ &+ \beta H(s-i_1) W^*_{(i_1+Q,i_2,i_3)}(y) + \delta_{i_2 S_1}\alpha_0 W^*_{(i_1,S_2,i_3)}(y) + \delta_{i_2 S_2}\eta_0 W^*_{(i_1,S_1,i_3)}(y) \\ &+ \sum_{k=1}^{J}(\delta_{i_2 S_1} r_k \mu_0) W^*_{(i_1-1,k,i_3)}(y) \end{aligned} \tag{10}$$

where

$$\begin{aligned} w_2 &= y + p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} + \bar{\delta}_{i_1 1}(i_1-1)\gamma + \beta H(s-i_1) + \delta_{i_2 S_1}\alpha_0 \\ &+ \delta_{i_2 S_2}\eta_0 + \sum_{k=1}^{J}(\delta_{i_2 S_1} r_k \mu_0). \end{aligned}$$

For $0 \le i_1 \le S, \quad 1 \le i_2 \le J, \quad 1 \le i_3 \le N-1,$

$$\begin{aligned} w_3 W^*_{(i_1,i_2,i_3)}(y) &= p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} W^*_{(i_1,i_2,i_3+1)}(y) + \bar{\delta}_{i_1 0} i_1\gamma W^*_{(i_1-1,i_2,i_3)}(y) \\ &+ \beta H(s-i_1) W^*_{(i_1+Q,i_2,i_3)}(y) + \sum_{k=1}^{J}(\delta_{i_2 k}\mu_k) W^*_{(i_1,S_0,i_3)}(y) + \sum_{k=1}^{J}(\delta_{i_2 k}\alpha_k) W^*_{(i_1,k,i_3)}(y) \end{aligned} \tag{11}$$

where

$$w_3 = y + p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} + \bar{\delta}_{i_1 0} i_1\gamma + \beta H(s-i_1) + \sum_{k=1}^{J}(\delta_{i_2 k}\mu_k) + \sum_{k=1}^{J}(\delta_{i_2 k}\alpha_k).$$

For $0 \le i_1 \le S, \quad J+1 \le i_2 \le 2J, \quad 1 \le i_3 \le N-1,$

$$\begin{aligned} w_4 W^*_{(i_1,i_2,i_3)}(y) &= p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} W^*_{(i_1,i_2,i_3+1)}(y) + \bar{\delta}_{i_1 0} i_1\gamma W^*_{(i_1-1,i_2,i_3)}(y) \\ &+ \beta H(s-i_1) W^*_{(i_1+Q,i_2,i_3)}(y) + \sum_{k=1}^{J}(\delta_{i_2 (J+k)}\eta_k) W^*_{(i_1,k,i_3)}(y) \end{aligned} \tag{12}$$

where

$$w_4 = y + p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} + \bar{\delta}_{i_10}i_1\gamma + \beta H(s-i_1) + \sum_{k=1}^{J}(\delta_{i_2(J+k)}\eta_k).$$

The system of linear equations can be utilized to obtain a recursive algorithm for calculating the moments for the waiting times. By differentiating the system of linear equations (4.2)–(4.5) a total of $(n+1)$ times and evaluating at $y=0$, the following is obtained.

For $0 \le i_1 \le S, \quad i_2 = S_0, \quad 1 \le i_3 \le N,$

$$\begin{aligned} w_5 E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] &- p(N-i_3)\lambda\bar{\delta}_{i_3N}\delta_{i_10}E\left[W^{(n+1)}_{(i_1,S_0,i_3+1)}\right] - (N-i_3)\lambda\bar{\delta}_{i_3N}\bar{\delta}_{i_10}E\left[W^{(n+1)}_{(i_1,S_1,i_3)}\right] \\ -i_1\gamma\bar{\delta}_{i_10}E\left[W^{(n+1)}_{(i_1-1,S_0,i_3)}\right] &- \beta H(s-i_1)E\left[W^{(n+1)}_{(i_1+Q,S_0,i_3)}\right] - (i_3-1)\theta\bar{\delta}_{i_10}E\left[W^{(n+1)}_{(i_1,S_1,i_3-1)}\right] \\ &= (n+1)E\left[W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \quad (13)$$

where

$$w_5 = p(N-i_3)\lambda\bar{\delta}_{i_3N}\delta_{i_10} + (N-i_3)\lambda\bar{\delta}_{i_3N}\bar{\delta}_{i_10} + i_1\gamma\bar{\delta}_{i_10} + \beta H(s-i_1) + i_3\theta\bar{\delta}_{i_10}.$$

For $1 \le i_1 \le S, \quad i_2 = S_1, S_2, \quad 1 \le i_3 \le N-1,$

$$\begin{aligned} w_6 E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] &- p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)}E\left[W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - \bar{\delta}_{i_11}(i_1-1)\gamma E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] \\ -\beta H(s-i_1)E\left[W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] &- \delta_{i_2S_1}\alpha_0 E\left[W^{(n+1)}_{(i_1,S_2,i_3)}\right] - \delta_{i_2S_2}\eta_0 E\left[W^{(n+1)}_{(i_1,S_1,i_3)}\right] \\ -\sum_{k=1}^{J}(\delta_{i_2S_1}r_k\mu_0)E\left[W^{(n+1)}_{(i_1-1,k,i_3)}\right] &= (n+1)E\left[W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \quad (14)$$

where

$$\begin{aligned} w_6 &= p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} + \bar{\delta}_{i_11}(i_1-1)\gamma + \beta H(s-i_1) + \delta_{i_2S_1}\alpha_0 \\ &\quad +\delta_{i_2S_2}\eta_0 + \sum_{k=1}^{J}(\delta_{i_2S_1}r_k\mu_0). \end{aligned}$$

For $0 \le i_1 \le S, \quad 1 \le i_2 \le J, \quad 1 \le i_3 \le N-1,$

$$\begin{aligned} w_7 E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] &- p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)}E\left[W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - \bar{\delta}_{i_10}i_1\gamma E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] \\ -\beta H(s-i_1)E\left[W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] &- \sum_{k=1}^{J}(\delta_{i_2k}\mu_k)E\left[W^{(n+1)}_{(i_1,S_0,i_3)}\right] - \sum_{k=1}^{J}(\delta_{i_2k}\alpha_k)E\left[W^{(n+1)}_{(i_1,k,i_3)}\right] \\ &= (n+1)E\left[W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \quad (15)$$

where

$$w_7 = p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} + \bar{\delta}_{i_10}i_1\gamma + \beta H(s-i_1) + \sum_{k=1}^{J}(\delta_{i_2k}\mu_k) + \sum_{k=1}^{J}(\delta_{i_2k}\alpha_k).$$

For $0 \leq i_1 \leq S, \quad J+1 \leq i_2 \leq 2J, \quad 1 \leq i_3 \leq N-1$,

$$w_8 E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] - p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} E\left[W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - i_1\gamma\bar{\delta}_{i_1 0} E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right]$$
$$-\beta H(s-i_1) E\left[W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] - \sum_{k=1}^{J}(\delta_{i_2(J+k)}\eta_k) E\left[{}^{(1)}W^{(n+1)}_{(i_1,k,i_3)}\right] = (n+1) E\left[W^{(n)}_{(i_1,i_2,i_3)}\right] \quad (16)$$

where

$$w_8 = p(N-(i_3+1))\lambda\bar{\delta}_{i_3(N-1)} + \bar{\delta}_{i_1 0} i_1\gamma + \beta H(s-i_1) + \sum_{k=1}^{J}(\delta_{i_2(J+k)}\eta_k).$$

Using the linear equations (4.6)–(4.9), the unknowns $E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right]$, $(i_1,i_2,i_3) \in E$ can be determined in terms of one order less. Since $E\left[W^{(n)}_{(i_1,i_2,i_3,)}\right] = 1$, when $n = 0$. The moments can be obtained up to a required order in a recursive way.
That is, for $n \geq 0$,

$$E[W^{(n)}] = \delta_{0n} + \left[\sum_{1_3=0}^{N-1} \phi^{(0,S_0,i_3)} E\left[W^{(n)}_{(0,S_0,i_3+1)}\right] + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-2} \phi^{(i_1,S_1,i_3+1)} E\left[W^{(n)}_{(i_1,S_1,i_3+1)}\right] \right. \quad (17)$$
$$\left. + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-2} \phi^{(i_1,S_2,i_3)} E\left[W^{(n)}_{(i_1,S_2,i_3+1)}\right] + \sum_{i_1=0}^{S}\sum_{i_2=1}^{2J}\sum_{i_3=0}^{N-2} \phi^{(i_1,i_2,i_3)} E\left[W^{(n)}_{(i_1,i_2,i_3+1)}\right]\right](1-\delta_{0n}),$$

which provides the n^{th} moments of the unconditional waiting time in terms of conditional moments of the same order.

5 System performance measures

In this section, some measures of system performance are derived in the steady state. Using this, the total expected cost rate is calculated.

5.1 Average on-hand inventory level

Let η_I denote the average on-hand inventory level in the steady state. Then

$$\eta_I = \sum_{i_1=1}^{S} i_1 \Pi^{(i_1)} \mathbf{e}.$$

5.2 Mean reorder rate

Let η_R denote the mean reorder rate in the steady state. A reorder is placed when the inventory level drops from $s+1$ to s. This may occur in the following cases:

- The server completes the essential service of a customer,
- any one of the $(s+1)$ items fails when the server is idle,
- any one of the s items fails either when the server is busy with FES or when the server is on interruption during FES providing,
- any one of the $(s+1)$ items fails either when the server is busy with Type k $(1 \leq k \leq J)$ service or when server is on interruption during Type k service providing.

Hence,

$$\eta_R = \sum_{i_3=0}^{N-1}\{(\sum_{k=0}^{J} r_k\mu_0) + s\gamma\}\pi^{(s+1,S_1,i_3)} + \sum_{i_3=0}^{N}(s+1)\gamma\pi^{(s+1,S_0,i_3)} + \sum_{i_3=0}^{N-1} s\gamma\pi^{(s+1,S_2,i_3)} + \sum_{i_2=1}^{2J}\sum_{i_3=0}^{N-1}(s+1)\gamma\pi^{(s+1,i_2,i_3)}.$$

5.3 Mean perishable rate

Let η_P denote the mean perishable rate in the steady state. Then

$$\eta_P = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N} i_1\gamma\pi^{(i_1,S_0,i_3)} + \sum_{i_1=1}^{S}\sum_{i=1}^{2}\sum_{i_3=0}^{N-1}(i_1-1)\gamma\pi^{(i_1,S_i,i_3)} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{2J}\sum_{i_3=0}^{N-1} i_1\gamma\pi^{(i_1,i_2,i_3)}.$$

5.4 Average number of customers in the orbit

Let η_{CO} denote the average number of customers in the steady state. Then

$$\eta_{CO} = \sum_{i_1=0}^{S}\sum_{i_3=1}^{N} i_3\pi^{(i_1,S_0,i_3)} + \sum_{i_1=1}^{S}\sum_{i=1}^{2}\sum_{i_3=1}^{N-1} i_3\pi^{(i_1,S_i,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=1}^{2J}\sum_{i_3=1}^{N-1} i_3\pi^{(i_1,i_2,i_3)}.$$

5.5 Average number of customers lost to the system

Let η_{CL} denote the average number of customers lost to the system in the steady state. Then

$$\eta_{CL} = \sum_{i_3=0}^{N-1}(1-p)(N-i_3)\lambda\pi^{(0,S_0,i_3)} + \sum_{i_1=1}^{S}\sum_{i=1}^{2}\sum_{i_3=0}^{N-2}(1-p)(N-(i_3+1))\lambda\pi^{(i_1,S_i,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=1}^{2J}\sum_{i_3=0}^{N-2}(1-p)(N-(i_3+1))\lambda\pi^{(i_1,i_2,i_3)}.$$

5.6 Effective interruption rate

Let η_{INTR} denote the effective interruption rate in the steady state. Then

$$\eta_{INTR} = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\alpha_0\pi^{(i_1,S_1,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=1}^{J}\sum_{i_3=0}^{N-1}\alpha_{i_2}\pi^{(i_1,i_2,i_3)}.$$

5.7 Effective repair rate

Let η_{RI} denote the effective repair rate in the steady state. Then

$$\eta_{RI} = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\eta_0\pi^{(i_1,S_2,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=J+1}^{2J}\sum_{i_3=0}^{N-1}\eta_{(i_2-J)}\pi^{(i_1,i_2,i_3)}.$$

5.8 The overall retrial rate

Let η_{OR} denote the overall rate of retrials in the steady state. Then

$$\eta_{OR} = \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}i_3\theta\pi^{(i_1,S_0,i_3)} + \sum_{i_1=1}^{S}\sum_{i=1}^{2}\sum_{i_3=1}^{N-1}i_3\theta\pi^{(i_1,S_i,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=1}^{2J}\sum_{i_3=1}^{N-1}i_3\theta\pi^{(i_1,i_2,i_3)}.$$

5.9 The successful retrial rate

Let η_{SR} denote the successful retrial rate in the steady state. Then

$$\eta_{SR} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}i_3\theta\pi^{(i_1,S_0,i_3)}.$$

5.10 The fraction of successful rate of retrial

Let η_{FR} denote the fraction of successful retrial rate in the steady state. Then

$$\eta_{FR} = \frac{\eta_{SR}}{\eta_{OR}}.$$

5.11 The probability that the server and system are idle

Let η_{PSI} denote the probability that server and system are idle is given by

$$\eta_{PSI} = \sum_{i_1=0}^{S}\pi^{(i_1,S_0,0)}.$$

5.12 The probability that the server is idle, but the system is not empty

Let η_{PSNI} denote the probability that server is idle and system is not empty is given by

$$\eta_{PSNI} = \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\pi^{(i_1,S_0,i_3)}.$$

5.13 Probability that the server is busy (FES or SOS)

Let η_{PSB} denote the probability that server is busy is given by

$$\eta_{PSB} = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\pi^{(i_1,S_1,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=1}^{J}\sum_{i_3=0}^{N-1}\pi^{(i_1,i_2,i_3)}.$$

5.14 Probability that the server is under repair

Let η_{POR} denote the probability that server is under repair is given by

$$\eta_{POR} = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\pi^{(i_1,S_2,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=J+1}^{2J}\sum_{i_3=0}^{N-1}\pi^{(i_1,i_2,i_3)}.$$

5.15 The effective rate at which arriving customers are lost on seeing an interrupted server

Let η_{POR} denote the effective rate at which arriving customers are lost on seeing an interrupted server is given by

$$\eta_{LI} = \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-2}(1-p)(N-(i_3+1))\lambda\pi^{(i_1,S_2,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=J+1}^{2J}\sum_{i_3=0}^{N-2}(1-p)(N-(i_3+1))\lambda\pi^{(i_1,i_2,i_3)}.$$

5.16 The effective rate at which arriving customers are lost when finding the inventory level as zero

Let η_{IL} denote the effective rate at which arriving customers are lost when finding the inventory level as zero is given by

$$\eta_{IL} = \sum_{i_3=0}^{N-1}(1-p)(N-i_3)\lambda\pi^{(0,S_0,i_3)}.$$

6 Cost analysis and sensitivity investigation

The expected total cost function per unit time is developed, wherein three decision variables S, s and N are considered. The objective is to find the optimum value of S, s and N, simultaneously so that the cost function is minimized. First, the following cost elements are defined: The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined so that

c_h is the inventory carrying cost per unit item per unit time,
c_s is the setup cost per order,
c_p is the perishable cost per unit item per unit time,
c_w is the waiting time cost of a customer in the orbit per unit time,
c_i is the cost per interruption per unit time,
c_r is the cost per repair per unit time,
c_l is the shortage cost of a customer per unit time.

Using cost values $c_h, c_s, c_p, c_w, c_i, c_r$ and c_l and the concept of crew-service equipment by White *et al.* [21], the total cost function $TC(S, s, N, J)$ per unit time in the steady state for this model is given by

$$TC(S, s, N, J) = c_h\eta_I + c_s\eta_R + c_p\eta_P + c_w\eta_{CO} + c_i\eta_{INTR} + c_r\eta_{PI} + c_l\eta_{CL}.$$

More clearly, $TC(S, s, N, J) =$

$$\begin{aligned}
& c_s \sum_{i_3=0}^{N-1} \{(\sum_{k=0}^{J} r_k\mu_0) + s\gamma\}\phi^{(s+1,S_1,i_3)} + c_s \sum_{i_3=0}^{N} (s+1)\gamma\phi^{(s+1,S_0,i_3)} + c_s \sum_{i_3=0}^{N-1} s\gamma\phi^{(s+1,S_2,i_3)} + \\
& c_s \sum_{i_2=1}^{2J} \sum_{i_3=0}^{N-1} (s+1)\gamma\phi^{(s+1,i_2,i_3)} + c_h \sum_{i_1=1}^{S} i_1\phi^{(i_1)}\mathbf{e} + c_p \sum_{i_1=1}^{S} \sum_{i_3=0}^{N} i_1\gamma\phi^{(i_1,S_0,i_3)} + \\
& c_p \sum_{i_1=1}^{S} \sum_{i=1}^{2} \sum_{i_3=0}^{N-1} (i_1-1)\gamma\phi^{(i_1,S_i,i_3)} + c_p \sum_{i_1=1}^{S} \sum_{i_2=1}^{2J} \sum_{i_3=0}^{N-1} i_1\gamma\phi^{(i_1,i_2,i_3)} + c_w \sum_{i_1=0}^{S} \sum_{i_3=1}^{N} i_3\phi^{(i_1,S_0,i_3)} + \\
& c_w \sum_{i_1=1}^{S} \sum_{i=1}^{2} \sum_{i_3=1}^{N-1} i_3\phi^{(i_1,S_i,i_3)} + c_w \sum_{i_1=0}^{S} \sum_{i_2=1}^{2J} \sum_{i_3=1}^{N-1} i_3\phi^{(i_1,i_2,i_3)} + c_i \sum_{i_1=1}^{S} \sum_{i_3=0}^{N-1} \alpha_0\phi^{(i_1,S_1,i_3)} + \\
& c_i \sum_{i_1=0}^{S} \sum_{i_2=1}^{J} \sum_{i_3=0}^{N-1} \alpha_{i_2}\phi^{(i_1,i_2,i_3)} + c_r \sum_{i_1=1}^{S} \sum_{i_3=0}^{N-1} \eta_0\phi^{(i_1,S_2,i_3)} + c_r \sum_{i_1=0}^{S} \sum_{i_2=J+1}^{2J} \sum_{i_3=0}^{N-1} \eta_{(i_2-J)}\phi^{(i_1,i_2,i_3)} + \\
& c_l \sum_{i_3=0}^{N-1} (1-p)(N-i_3)\lambda\phi^{(0,S_0,i_3)} + c_l \sum_{i_1=1}^{S} \sum_{i=1}^{2} \sum_{i_3=0}^{N-2} (1-p)(N-i_3)\lambda\phi^{(i_1,S_i,i_3)} + \\
& c_l \sum_{i_1=0}^{S} \sum_{i_2=1}^{2J} \sum_{i_3=0}^{N-2} (1-p)(N-i_3)\lambda\phi^{(i_1,i_2,i_3)}.
\end{aligned}$$

A sensitivity investigation is given by considering the following parameters and cost values: $\lambda = 0.04,\ \beta = 5,\ \gamma = 4,\ \theta = 3,\ \mu_0 = 8,\ \mu_1 = 0.17,\ \mu_2 = 0.06,\ \mu_3 = 16,\ \alpha_0 = 0.05,\ \alpha_1 = 0.08,\ \alpha_2 = 0.5,\ \alpha_3 = 0.7,\ \eta_0 = 4,\ \eta_1 = 11,\ \eta_2 = 6,\ \eta_3 = 0.06,\ r_0 = 0.25,\ r_1 = 0.25,\ r_2 = 0.25,\ r_3 = 0.25,\ c_h = 0.35,\ c_s = 3,\ c_p = 0.3,\ c_w = 2,\ c_l = 5, c_i = 4, c_r = 12.$ The performance of the expected total cost function is investigated by allowing for any two variables to change, while fixing others at a constant level. Figure 2 reveals the 3-dimensional plot of $TC(S, s)$. In Tables 1–3, show the cost function of $TC(S, s, 10, 3)$,

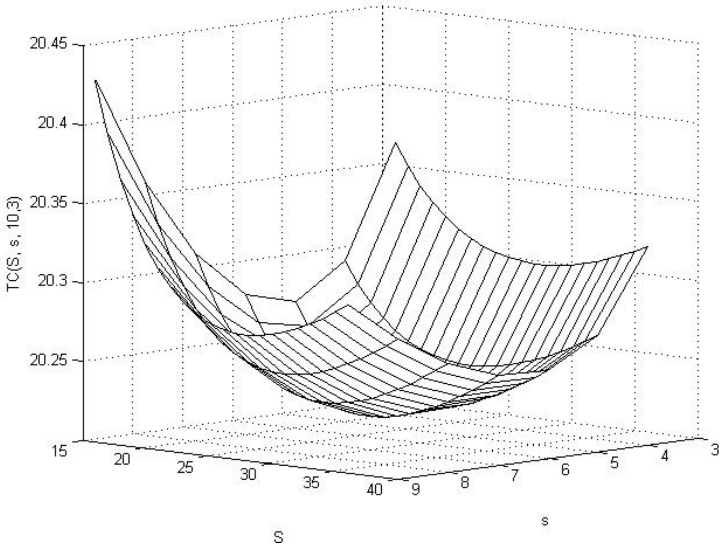

Figure 2: A three-dimensional plot of the cost function.

$TC(50, s, N, 3)$ and $TC(S, 6, N, 3)$ by fixed parameters and cost values as constant. After getting the Local Optima (LO) S^*, s^* and N^*, the sensitivity investigation is carried out to observe how the changes in S, s and N affect the cost function. In Tables 1–3, the values displayed in **bold** is the smallest value amongst the values in that row and likewise the values that are *underlined* is the smallest value amongst the values in that column. Therefore, a value (bold and underlined) shows a local minimum of TC. More clearly, varying S and s, the optimal values are $S^* = 30$, $s^* = 6$ and $TC^* = 20.213022$; varying s and N, the optimal values are $s^* = 8$, $N^* = 14$ and $TC^* = 9.264222$; varying S and N, the optimal values are $S^* = 34$, $N^* = 12$ and $TC^* = 7.821037$.

Next, a numerical illustration is performed to the optimal values (S^*, s^* and TC^*) based on changes in particular cost values. The numerical results are presented by considering the different cost values of c_h, c_s, c_p, c_w, c_l, c_i and c_r. The following are observed from Tables 4–10.

1. The optimal cost increases, when c_s, c_h, c_p, c_l, c_i, c_r and c_w increase. The optimal cost is more sensitive to c_w than to c_s, c_h, c_p, c_i, c_r and c_l.
2. As c_h increases, as expected, the optimal values s^* and S^* decrease monotonically. This is expected since the holding cost increases, low stock is maintained in the inventory.
3. When c_l and c_w increase, the optimal values s^* and S^* increase monotonically. This is because if the waiting cost and shortage cost of a customer increase then high inventory is maintained to reduce the number of waiting (lost) customers.
4. If the setup cost c_s increases, it is a common decision that more stock has to be maintained to avoid frequent ordering. This fact is also observed in the model.
5. It is noted that, if c_i and c_r increase, the optimal values of s^* and S^* increase monotonically and when the perishable cost c_p increases, s^* and S^* decrease monotonically.

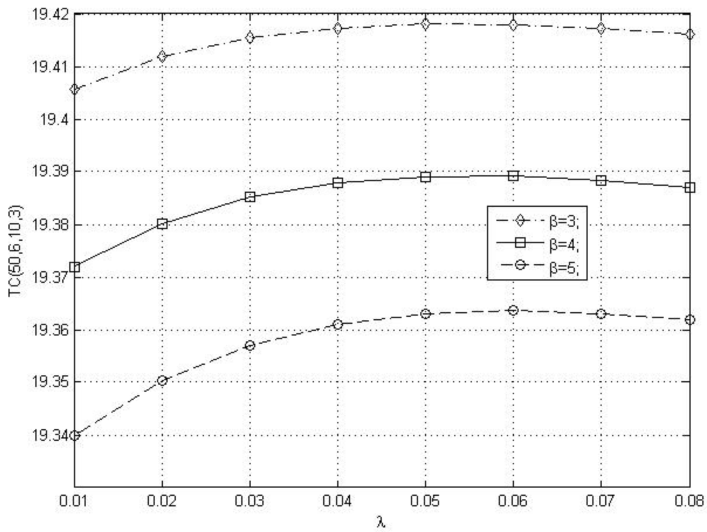

Figure 3: λ vs β on TC

Now, the impact of the parameter values λ, γ, β, θ, μ_0 μ_1, μ_2 and μ_3 on the total expected cost rate are looked at. From Figures 3–8, the following is observed:

1. The optimal expected cost rate increases when λ and γ increase.
2. The optimal expected cost rate decreases when β, μ_0, μ_1, μ_2, μ_3 and θ increase.

The numerical illustrations are obtained by considering different service cases as follows:

Case 1: $r_0 = 0.25, r_1 = 0.25, r_2 = 0.25, r_3 = 0.25$;
Case 2: $r_0 = 0.4, r_1 = 0.2, r_2 = 0.2, r_3 = 0.2$;
Case 3: $r_0 = 0.2, r_1 = 0.4, r_2 = 0.2, r_3 = 0.2$;
Case 4: $r_0 = 0.2, r_1 = 0.2, r_2 = 0.4, r_3 = 0.2$;
Case 5: $r_0 = 0.2, r_1 = 0.2, r_2 = 0.2, r_3 = 0.4$;
Case 6: $r_0 = 1, r_1 = 0, r_2 = 0, r_3 = 0$;
Case 7: $r_0 = 0.5, r_1 = 0.5, r_2 = 0, r_3 = 0$.

Now, the impact of parameter values λ, β, θ and different service cases (Cases 1–7) on TC are looked at. From Figures 9–11, the following is observed:

1. From Figures 9 and 10:

   $$TC_{\text{class }4} > TC_{\text{class }1} > TC_{\text{class }3} > TC_{\text{class }2} > TC_{\text{class }5} > TC_{\text{class }7} > TC_{\text{class }6}.$$

2. From Figure 11:

   $$TC_{\text{class }4} > TC_{\text{class }1} > TC_{\text{class }2} > TC_{\text{class }3} > TC_{\text{class }5} > TC_{\text{class }7} > TC_{\text{class }6}.$$

S \ s	4	5	6	7	8
28	20.238179	20.214607	**20.214144**	20.228737	20.254321
29	20.237064	<u>20.213788</u>	**20.213090**	20.226990	20.251429
30	<u>20.236783</u>	20.213858	<u>**20.213022**</u>	<u>20.226369</u>	20.249854
31	20.237253	20.214724	**20.213832**	20.226743	<u>20.249435</u>
32	20.238399	20.216305	**20.215426**	20.228002	20.250036

Table 1: Total expected cost rate as a function of S and s

N \ s	6	7	8	9	10
12	9.475022	9.375660	**9.264644**	9.344231	9.356036
13	9.474643	9.375323	**9.264346**	9.343971	9.355811
14	9.474447	9.375164	<u>**9.264222**</u>	<u>9.343879</u>	9.355750
15	<u>9.474402</u>	<u>9.375149</u>	**9.264237**	9.343923	<u>9.355822</u>
16	9.474479	9.375255	**9.264369**	9.344080	9.356003

Table 2: Total expected cost rate as a function of s and N

S \ N	10	11	12	13	14
32	7.900699	7.847420	**7.827450**	7.834535	7.869472
33	7.882873	7.836107	**7.822051**	<u>7.834425</u>	7.868038
34	7.870182	7.829554	<u>**7.821037**</u>	7.838337	<u>7.866280</u>
35	<u>7.862038</u>	<u>7.827200</u>	**7.823878**	7.845770	7.877730
63	7.877938	**7.828566**	7.830121	7.856304	7.891997

Table 3: Total expected cost rate as a function of S and N

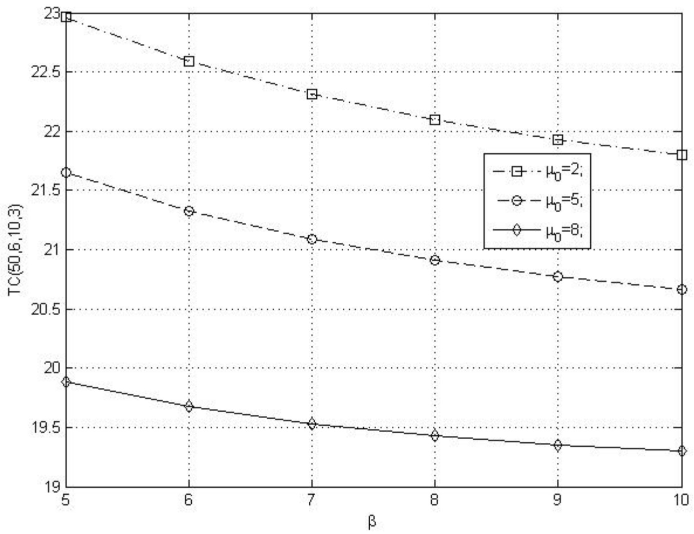

Figure 4: β vs μ_0 on TC

c_s / c_h	1.5		2		2.5		3		3.5	
0.25	50	6	56	6	59	5	62	4	65	4
	21.365796		21.534480		21.690773		21.837029		21.968881	
0.30	47	6	51	6	56	5	60	4	63	4
	21.378759		21.546029		21.701847		21.847088		21.978081	
0.35	45	5	48	5	51	5	56	4	60	4
	21.391656		21.557579		21.712886		21.857099		21.988377	
0.40	40	5	44	5	48	4	51	3	56	3
	21.404441		21.569125		21.723925		21.867110		21.998115	
0.45	36	4	40	4	44	4	48	3	52	3
	21.417227		21.580540		21.734964		21.877121		22.007852	

Table 4: Variation in optimal values for different values of c_h and c_s with $c_p = 0.3,\ c_w = 2,\ c_l = 5,\ c_i = 4,\ c_r = 12$.

c_p / c_h	0.1		0.2		0.3		0.4		0.5	
0.25	50	6	48	5	45	5	40	4	35	3
	21.365796		22.257257		22.919692		23.487756		23.988367	
0.30	47	6	46	5	40	5	36	4	33	3
	21.378759		22.273638		22.937827		23.508647		24.007861	
0.35	45	5	40	4	35	4	32	4	30	3
	21.391656		22.289913		22.955962		22.528934		24.027356	
0.40	40	5	36	4	30	4	27	3	24	3
	21.404441		22.306188		22.973757		23.549029		24.046851	
0.45	36	4	30	4	24	4	20	3	18	3
	21.417227		22.321962		22.991351		23.569123		24.066346	

Table 5: Variation in optimal values for different values of c_h and c_p with $c_s = 3,\ c_w = 2,\ c_l = 5,\ c_i = 4, c_r = 12$.

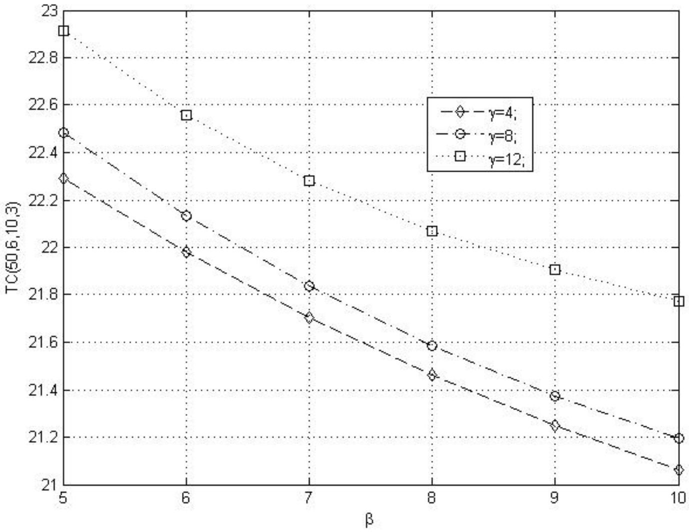

Figure 5: β vs γ on TC.

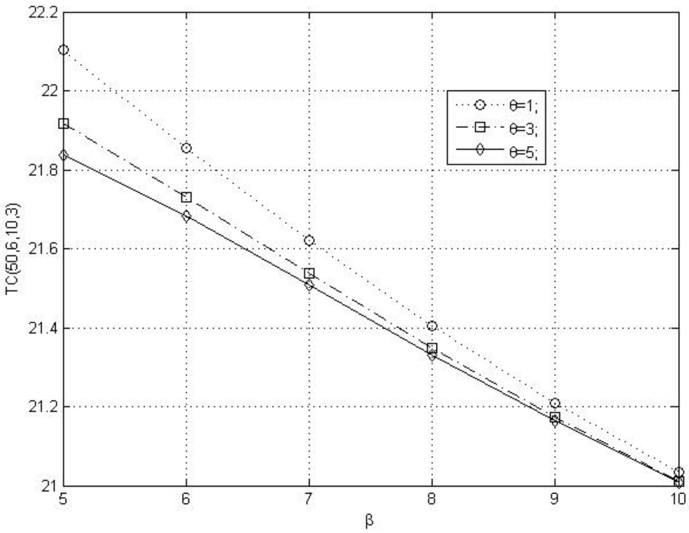

Figure 6: β vs θ on TC.

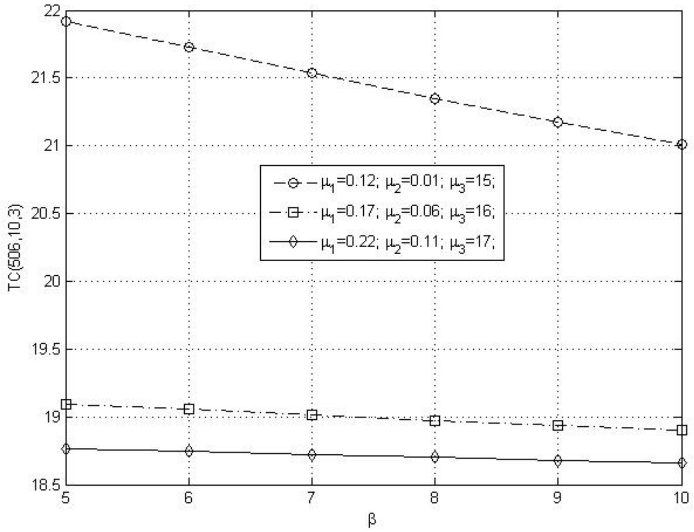

Figure 7: β vs μ_1, μ_2, μ_3 on TC.

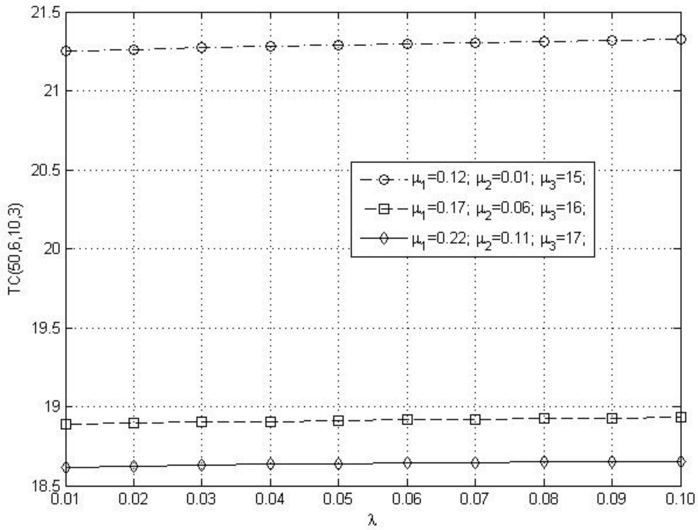

Figure 8: β vs μ_1, μ_2, μ_3 on TC.

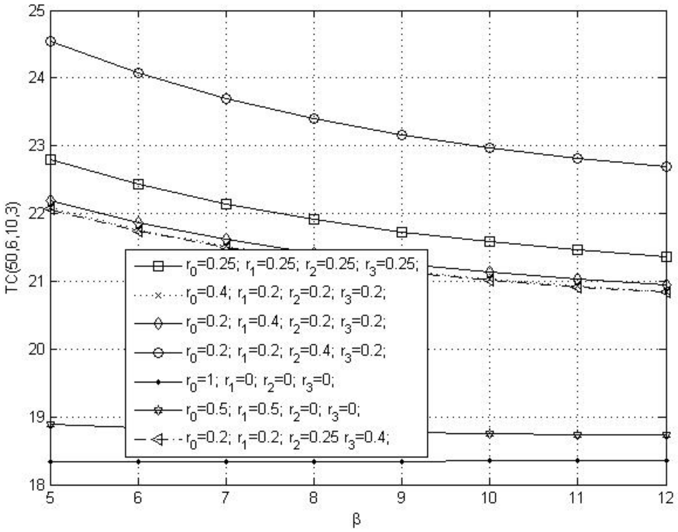

Figure 9: β vs service cases on TC.

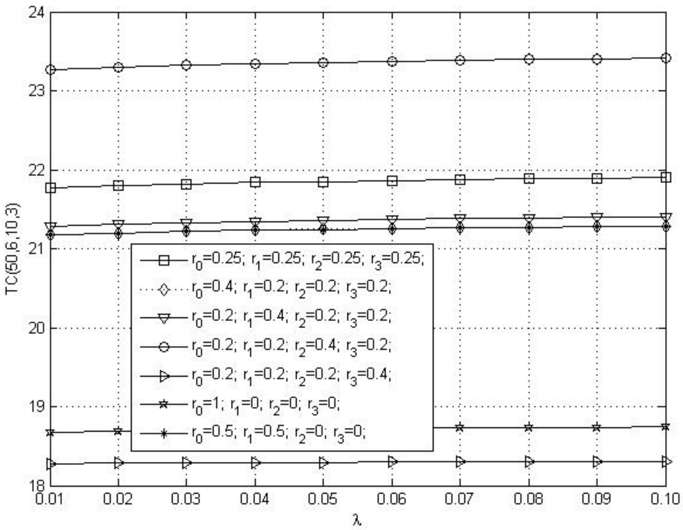

Figure 10: β vs service cases on TC.

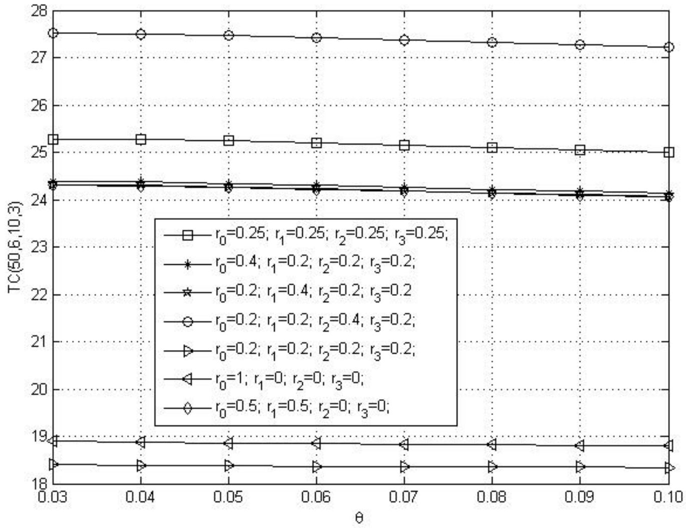

Figure 11: β vs service cases on TC.

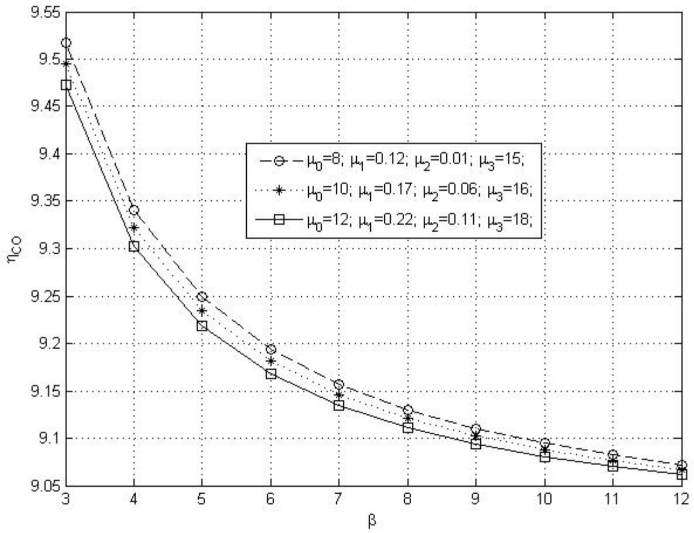

Figure 12: β vs μ_0, μ_1, μ_2 and μ_3 on η_{CO}.

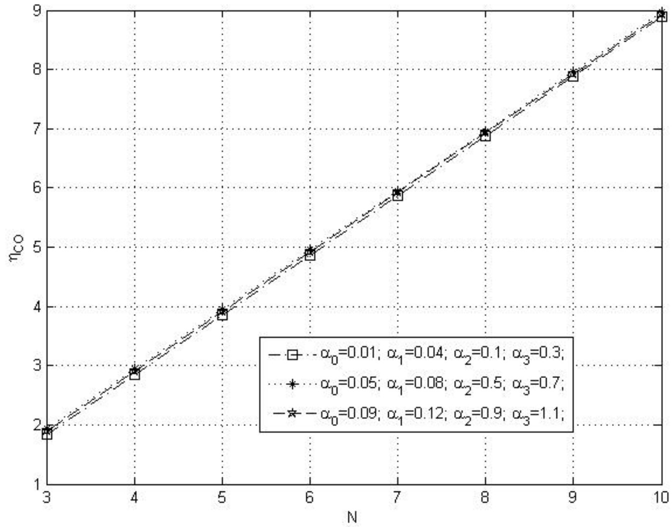

Figure 13: N vs $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ on η_{CO}.

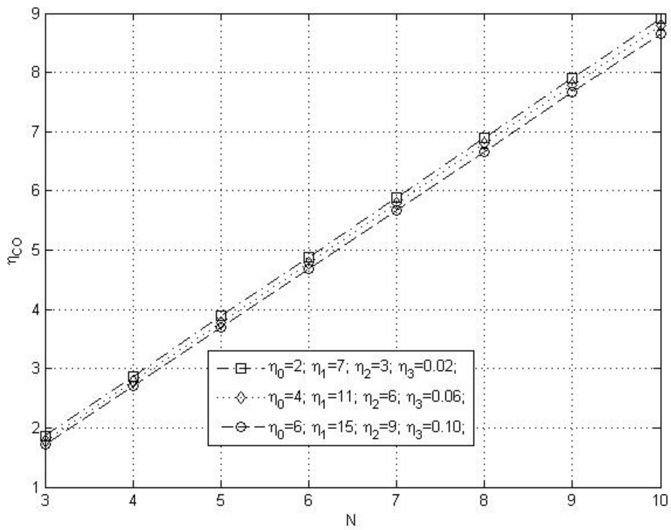

Figure 14: N vs $\eta_0, \eta_1, \eta_2, \eta_3$ on η_{CO}.

c_w / c_h	2		3		4		5		6	
0.25	50	6	54	6	55	7	56	8	57	8
	21.365796		30.361160		39.356524		48.351888		57.347252	
0.30	47	6	50	6	52	7	53	7	54	7
	21.378759		30.374123		39.369487		48.364851		57.360215	
0.35	45	5	48	5	50	6	51	6	53	7
	21.391656		30.387039		39.382423		48.377807		57.373178	
0.40	40	5	45	5	48	5	49	6	50	6
	21.404441		30.399825		39.395209		48.390593		57.385976	
0.45	36	4	40	4	45	5	49	5	50	6
	21.417227		30.412611		39.407995		48.403379		57.398762	

Table 6: Variation in optimal values for different values of c_h and c_w with $c_s = 3,\ c_p = 0.3,\ c_l = 5,\ c_i = 4,\ c_r = 12$.

c_p / c_s	0.1		0.2		0.3		0.4		0.5	
1.5	50	6	48	5	45	5	40	4	35	3
	21.365796		22.257257		22.919692		23.487756		23.988367	
2	54	4	52	4	48	4	45	3	40	3
	21.534480		22.457181		23.164813		23.738541		24.235497	
2.5	57	4	55	4	52	3	50	3	47	2
	21.690773		22.642965		23.371365		23.953040		24.471896	
3	60	3	57	3	55	3	52	2	50	2
	21.837029		22.821527		23.554849		24.159996		24.699138	
3.5	63	3	60	3	59	3	57	2	53	2
	21.968881		22.993137		23.733211		24.360297		24.918487	

Table 7: Variation in optimal values for different values of c_s and c_p with $c_h = 0.35,\ c_w = 2,\ c_l = 5, c_i = 4, c_r = 12$.

c_w / c_s	2		3		4		5		6	
1.5	50	6	54	6	55	7	56	8	57	8
	21.365796		30.361160		39.356524		48.351888		57.347252	
2	54	4	54	5	55	6	56	7	57	8
	21.534480		30.529822		39.525163		48.520505		57.515847	
2.5	57	4	57	5	58	6	60	6	62	7
	21.690773		30.686169		39.681566		48.676963		57.672359	
3	60	3	60	4	62	5	64	5	65	4
	21.837029		30.832368		39.827707		48.823046		57.818385	
3.5	63	3	64	3	65	4	66	4	67	4
	21.968881		30.964270		39.959660		48.955049		57.950439	

Table 8: Variation in optimal values for different values of c_s and c_w with $c_h = 0.35,\ c_p = 0.3,\ c_l = 5, c_i = 4, c_r = 12$.

c_s \ c_l	5		6		7		8		9	
1.5	42	5	47	6	50	6	52	7	55	8
	21.273175		21.319551		21.365796		21.412040		21.458179	
2	45	4	50	4	54	4	58	6	60	7
	21.441252		21.487907		21.534480		21.580947		21.627414	
2.5	49	4	53	4	57	4	60	5	64	6
	21.598933		21.644853		21.690773		21.736552		21.782295	
3	54	3	59	3	60	3	64	4	69	5
	21.744041		21.790535		21.837029		21.883401		21.929725	
3.5	58	3	61	3	63	3	66	3	72	4
	21.876897		21.922889		21.968881		22.014732		22.060561	

Table 9: Variation in optimal values for different values of c_s and c_l with $c_h = 0.35,\ c_p = 0.3,\ c_w = 2, c_i = 4, c_r = 12$.

c_i \ c_r	3		4		5		6		7	
10	42	5	44	6	45	6	46	6	47	7
	21.205227		21.365796		21.522320		21.678341		21.833571	
12	45	6	46	6	47	6	48	7	49	7
	21.347984		21.504086		21.659696		21.810874		21.959569	
14	53	6	54	7	55	7	56	7	57	7
	21.485517		21.639196		21.787930		21.936342		22.084391	
16	55	7	56	7	56	7	57	7	58	8
	21.616291		21.764736		21.912784		22.060517		22.203573	
18	56	7	57	7	58	8	59	8	60	8
	21.741181		21.888943		22.033663		22.176304		22.318753	

Table 10: Variation in optimal values for different values of c_r and c_i with $c_h = 0.35,\ c_s = 3,\ c_p = 0.3,\ c_w = 2,\ c_l = 5$.

Finally, the impact of the parameters λ, β, μ_0, μ_1, μ_2, μ_3, η_0, η_1, η_2, η_3, α_0, α_1, α_2, α_3 and N on the expected number of customers in orbit η_{CO} are studied. The following is observed from Figures 12–14.

1. If β, μ_i and η_i, $i = 0, 1, 2, 3$ increase, then the expected number of customers in the orbit η_{CO} decreases.
2. When N and α_i, $i = 0, 1, 2, 3$, increase, then η_{CO} increases.

7 Conclusion

In this paper, a continuous review stochastic inventory system with J additional options for service, server interruptions, returning customers and finite populations were analyzed. The stationary distribution of the number of customers in the orbit and the inventory level is obtained by matrix method. Various system performance measures are derived and the long-run total expected cost rate is calculated and also the Laplace-Stieltjes transform of the waiting time of the orbiting customer is derived. By assuming a suitable cost structure on the inventory system, extensive numerical illustrations were presented to show the effect of change of values for constants on the total expected cost rate.

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