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A MATHEMATICAL MODEL FOR RESIDENTIAL PLANNING IN RICHARDS BAY

by

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ABSTRACT

The formulation of a systems dynamics model which was applied to obtain forecasts of important urban variables such as population and housing, is discussed. It is shown that the model simulated the growth trends in the town, at least for the period for which data was available, satisfactorily. A sensitivity analysis of the model was carried out and no sensitive parameters were identified during the 6 year simulation interval. An optimization strategy by which the occupation rate of housing was restricted to certain limits, is also discussed.

1. INTRODUCTION

The development of a urban model by Joubert [1] at the department of Applied Mathematics at the University of Zululand came to the attention of the Town Board of Richards Bay during 1982. This led to a request from the Town Board for a mathematical model which could be used for short term forecasting of the values of certain dynamic urban variables such as population, housing, etc.

The most urgent problem facing the town planning department of the Town Board was the timeous proclamation and development of new residential areas. Proclamation of new residential areas involves large amounts of money and if it takes place prematurely (i.e. at a time when no urgent demand for residential land exists) the situation might arise that these funds could have been applied more efficiently elsewhere in the town. If, on the other hand, the development is delayed for too long the shortage of residential land may become a limiting factor in the growth processes of the town. The town planning department decided to solve this problem by implementing a mathematical model which could be used to forecast the housing needs for a 6 year period.

A large scale mathematical model with 23 state variables had already been developed by Joubert [1] to simulate growth in Richards Bay. This model was however not calibrated to follow historical growth patterns and furthermore it also contained a large body of information which was irrelevant to the specific problem of housing demand. It was consequently decided to develop a simpler and goal-orientated model which could be used to solve the problem posed above.

2. THE HOUSING MODEL

This new model consisted of 5 state variables which represented the most important relevant urban variables. These state variables are defined by a coupled system of first order,

THE RICHARDS BAY HOUSING MODEL

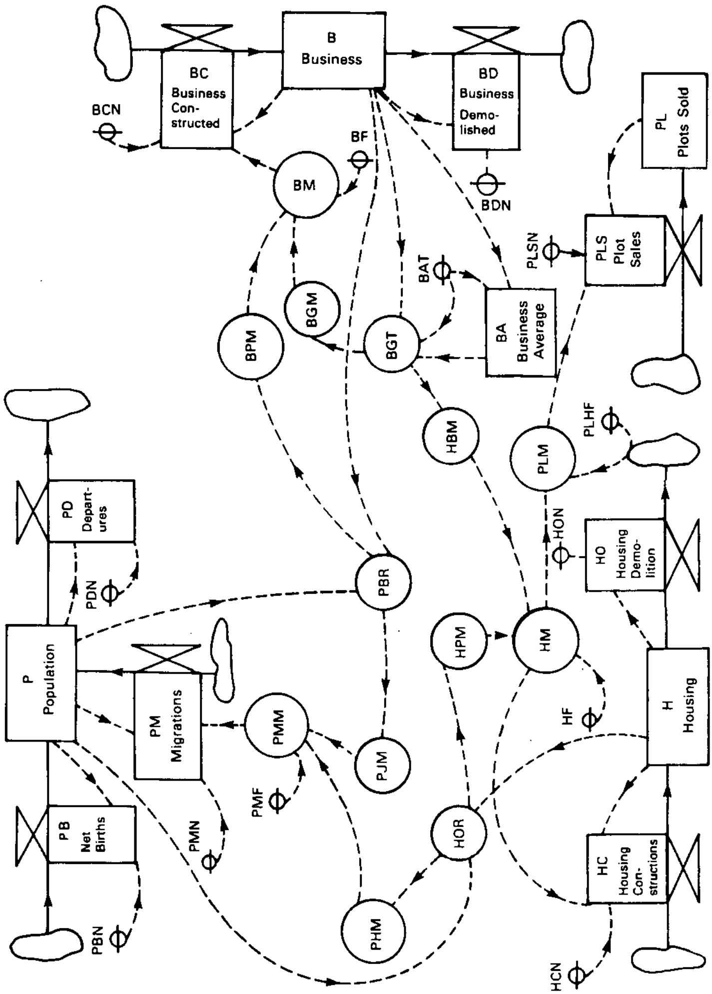

FIGURE 1

ordinary, non-linear differential equations. The system consisting of 5 simultaneous differential equations was numerically solved using an Euler integration algorithm. The time dependent solution was found for the 5 year period from January 1982 to January 1987.

A system dynamics flow chart diagram (see Forrester [2] for a description of system dynamics) for the system under discussion is shown in figure 1. The state variables are represented by rectangles, the rate variables by valve symbols and the auxilliary variables by circles. Constant parameters are represented by the symbol $-\!\ominus\!-$. The formulation of the model equations and relations between variables can be followed on the flow chart diagram.

A complete list of all model equations as well as all symbol names appear in the appendix. The most important model variable is the number of houses (H) and consequently only equations directly involved in its calculation are discussed in some detail.

The number of houses (H) at any time in the town is a state variable and is defined by the following differential equation :

$$\frac{dH}{dt} = HC - HO \qquad \text{.......} \quad (1)$$

H = Number of houses in town (Houses)
HC = Number of houses constructed per year (Houses/Year)
HO = Number of houses demolished per year (Houses/Year)
t = Time (Year)

$$HC = H * HCN * HM \qquad \text{.......} \quad (2)$$

HC = Number of houses constructed per year (Houses/Year)
H = Number of houses in town (Houses)
HM = Housing multiplier (Dimensionless)
HCN = Normal housing construction rate (Fraction/Year)
HCN = 0,135

Equation (2) is an example of the calculation of a typical rate variable in system dynamics. The rate variable (HC) is defined as the product of a state variable (H), a normal rate (HCN), and a dimensionless multiplier (HM) which adjusts the normal rate. It is further assumed that the rate at which houses are constructed (HC) is a fraction of the number of houses (H) in town at any time. The value of 0,135 for HCN implies that under normal circumstances (i.e. when HM = 1) houses are constructed at an annual rate of 13,5 percent. This value was evaluated from available data supplied by the Town Board.

HM = HF * HPM * HBM .... (3)

HM	= Housing multiplier	(Dimensionless)
HPM	= Housing population multiplier	(Dimensionless)
HBM	= Housing business multiplier	(Dimensionless)
HF	= Housing factor	(Dimensionless)
HF	= 1,0	

The parameter HF(= 1) is included in equation (3) for two reasons. During a calibration of the model HF may be adjusted in order to cause certain model variables to fit time series data. This parameter is also important in a sensitivity analysis of the model. The effect of changes in it can be interpreted as the sensitivity of the model to changes in the other factors, HPM and HBM, of the term on the right hand side of equation 3. (See for example [1] Chapter 4,for a discussion of table function sensitivity.)

Equation (3) further implies that the housing multiplier (HM) and by equation (2) also the annual housing constructions (HC), depend on a population factor (HPM) and a business factor (HBM). Equations (4) and (5) respectively express HPM and HBM as functions of the occupancy rate of housing (HOR) and the business growth rate (BGT). The variable HOR is later defined in equation (9) while the definition of BGT is listed together with the other model equations in the appendix. It should be noted that both HOR and BGT may be expressed in terms of state variables and (constant) parameters.

HPM = HPM(HOR) .... (4)

HPM = Housing population multiplier (Dimensionless)
HOR = Housing occupancy rate (People/house)

HBM = HBM(BGT) .... (5)

HBM = Housing business multiplier (Dimensionless)
BGT = Business growth rate (Fraction/year)

The functional relations between HPM and HOR and HBM and BGT respectively appear in the appendix in the form of so called table functions. These functions are specified as a set of co-ordinates and linear interpolation between these points is assumed. The shape of a specific table function may either be determined from available data or in the absence of data may be assumed. Such assumptions can have significant influence on model behaviour and should, as was done in this case, be thoroughly investigated by means of a sensitivity analysis.

Equation (6) defines the demolition rate (HO) of houses in town. According to Alfeld and Graham [3] it may be assumed that the average life span of a house is in the order of 90 years. This implies that during the 5 year simulation period early in the town's history, only a few houses deteriorate to such an extent that it becomes necessary to be demolished. A relatively low normal demolition rate (HON) of 0,01 is consequently assumed.

HO = H * HON .... (6)

HO = Number of houses demolished per year (Houses/year)
H = Number of houses in town (Houses)
HON = Normal housing demolition rate (Fraction/year)
HON = 0,01

From the equations discussed as well as those listed in the appendix it follows that state variables are defined by equations of the following form :

$$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, \underline{p}, t) \qquad \ldots\ldots\ldots \quad (7)$$

$\underline{x}$ = state vector

$\underline{p}$ = parameter vector

t = time

3. RESULTS

Before any results could be obtained, the model had to be calibrated by means of historical data made available by the Town Board. It was also necessary to estimate some parameters on which no data was available. An example of such an estimated parameter is the normal demolition rate (HON) of houses.

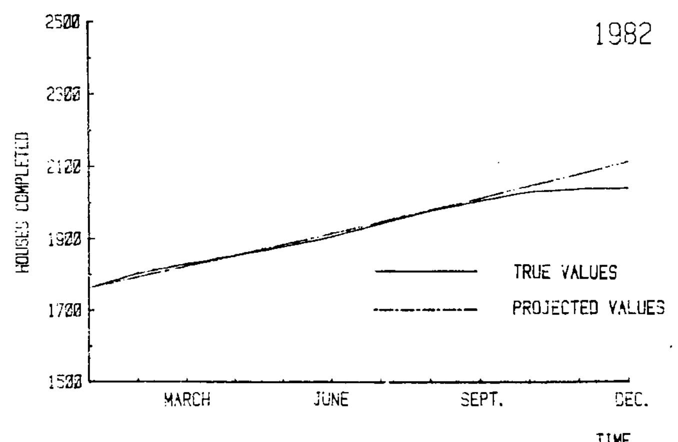

FIGURE 2

Figure 2 shows the forecasted monthly house completions and on the same axis system the actual number of houses completed for the period January to December 1982.

These curves do not differ significantly at any time during 1983 and it may be concluded that the model could provide a sound basis for short term forecasting of housing demand. Similar results were found for population forecasts during the abovementioned period.

4. SENSITIVITY OF THE MODEL

The importance of sensitivity analyses in the modelling process has been emphasised by Vermeulen and De Jongh [4]. A parameter sensitivity analysis involving the following 9 parameters was carried out on the housing model, EVN, EHF, HCN, HF, PMN, BCN, BF, PDN, BON. (The names of these parameters appear in the appendix.) The model exhibited small values for all normalised sensitivity functions which in general are of the following form :

$$N_{i,j}(t) = \frac{\partial x_i(t)}{\partial p_j} \Big/ \frac{x_i(t)}{p_j} \qquad \ldots\ldots\ (8)$$

x_i = ith state variables

p_j = jth parameter

Perturbations of 1% in each of the 9 parameters were separately considered and in none of the cases did any of these perturbations cause changes greater than 1% in any of the state variables during the 5 year simulation interval. A combination of simultaneous perturbations of 10% each in the 3 most sensitive parameters(HCN, PMN and HF) were consequently considered. This resulted in increments of 15,9% in the housing (H), 10,1% in the population (P) and 14,8% in the number of plots sold (PL) at the end of the simulation period. The influence of this combined parameter perturbation on the number of houses (H) in the town is shown in figure 3.

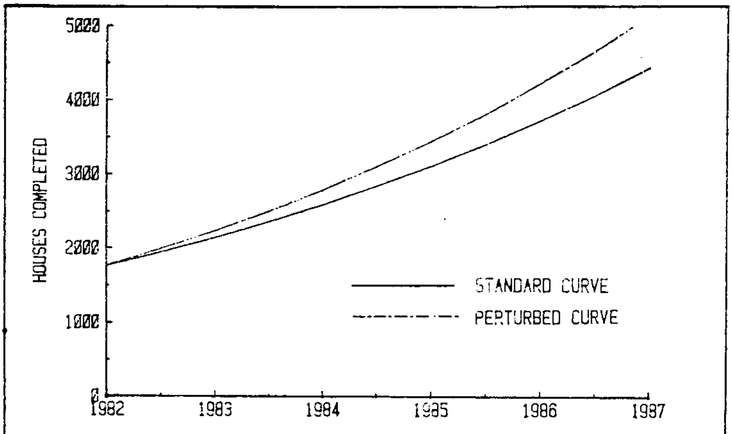

FIGURE 3

5. OPTIMIZATION OF THE OCCUPANCY RATE (HOR)

The occupancy rate of housing at the start of the simulation interval was HOR = 5,42 . This value was slightly high by urban standards and it was decided to investigate the effects of a control on this variable on the simulated values of some other model variables. This exercise was not part of the Richards Bay Town Board's request but is included to illustrate how an objective function, such as HOR, can be controlled in a model of this nature.

The control consisted of stepwise increments in the parameters HCN and PMN that directly determine HOR. Consider the definition of HOR :

$$HOR = P/H \qquad \text{.......} \quad (9)$$

HOR =	Housing occupancy rate	(People/house)
P =	Number of people in town	(People)
H =	Number of houses in town	(Houses)

From equations (1) and (2) follows that increments in the constant parameter HCN leads to an increased rate of housing construction. In a similar way an increment in the normal migration rate (PMN) leads to increased migrations to the town.

The control was designed to become operative whenever HOR exceeded the limits of a desired interval [4,5 ; 5,0]. In cases where HOR > 5,0 , HCN was increased by 10% and PMN simultaneously decreased by 1%, until HOR once more entered the prescribed interval. The numerical values of the increments of 10% and 1% respectively were arbitrarily chosen to reflect the relative ease by which these parameters can be adjusted in a real life situation. Conversely PMN was increased by 1% and HCN decreased by 10% whenever HOR < 4,5 .

The behaviour of the objective function, HOR, over the simulation period with and without the optimization algorithm in operation is shown in figure 4.

The percentage deviation of the housing (H) and population (P) from the standard values due to the optimization algorithm appear in table 1. These deviations are given at yearly intervals during the simulation period. It follows from table 1 that no large deviations resulted from this particular optimization algorithm.

6. CONCLUSION

An important achievement of this project was the fact that urban decision-makers made use of a mathematical model in the formulation of land proclamation policy. The systems dynamics model described the system under discussion adequately without being particularly sensitive to a wide range of parameter perturbations.

TABLE 1

Percentage Deviation from standard values of Housing and Population due to optimization strategy - 1982 - 1987

	Housing Deviation	Population Deviation
1982	-	-
1983	13,2	0,4
1984	23,9	4,5
1985	14,5	8,1
1986	5,6	8,4
1987	6,1	6,8

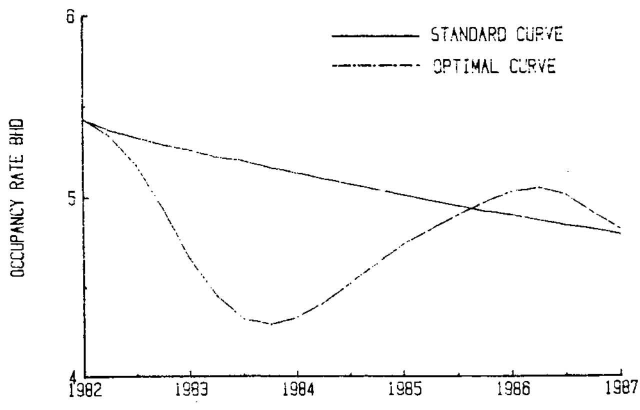

FIGURE 4

It is of interest that the Town Board of Richards Bay subsequently requested a continuation of this project. Data on housing and plot sales become available at monthly intervals, and it was felt that six monthly updates of the projections would enable them to become aware of any possible future shortfalls in availability of residential land. The town board further created a post for an urban researcher who could assist with the task of data collection and preparation. As part of the project it was also envisaged that town board staff could be trained to run and experiment with the model on the town board's own computer.

These arrangements are, from a modeller's point of view, very satisfying as close collaboration between modeller and user ensures constant feedback regarding the model's performance. Specifically, this feedback led to a few refinements of the model, so that the one presently in use differs slightly from the model described in text.

The algorithm employed to restrict the chosen objective function's values to a given interval can readily be extended to accommodate more complicated objective functions. Optimization techniques such as the one described above may have important practical implications for urban decision-makers. It is further foreseen that the incorporation of these techniques in a mathematical model may contribute towards the better understanding of the system's behaviour in response to particular urban strategies.

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APPENDIX

Model Equations, Symbol Names and State variable values

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---------------------------------------------------------------------
RICHARDS BAY PROJECT
---------------------------------------------------------------------
TABLE FUNCTIONS USED IN HOUSING MODEL
---------------------------------------------------------------------
POPULATION HOUSING FUNCTION
FUNCTION PHMT(HOR)
DATA X1/2.5,3.3,5.3,5.5/
DATA Y1/1.4,1.3,0.7,0.6/

POPULATION JOBS FUNCTION
FUNCTION PJMT(PBR)
DATA X1/40.0,43.0,63.0,70.0/
DATA Y1/1.4,1.3,0.7,0.6/

HOUSING POPULATION FUNCTION
FUNCTION HPMT(HOR)
DATA X1/2.5,3.3,5.3,5.5/
DATA Y1/0.6,0.7,1.3,1.4/

HOUSING BUSINESS FUNCTION
FUNCTION HBMT(BGT)
DATA X1/-1.5,1.5/
DATA Y1/0.4,1.6/

BUSINESS POPULATION FUNCTION
FUNCTION BPMT(PBR)
DATA X1/40.0,43.0,63.0,70.0/
DATA Y1/0.6,0.7,1.3,1.4/

BUSINESS GROWTH FUNCTION
FUNCTION BGMT(BGT)
DATA X1/-1.5,1.5/
DATA Y1/0.6,1.4/

---------------------------------------------------------------------
SPECIFICATION OF PARAMETERS AND INITIAL VALUES
---------------------------------------------------------------------
PLSN=0.145,PLHF=1.0,HCN=0.135,HF=1.0,HON=0.001,PBN=0.005,
PMN=0.16,PMF=1.0,PDN=0.01,BCN=0.14,BF=1.0,BON=0.025,BAT=4.0,
HI=1764.0,PI=9571.0,BI=324.0,BAI=324.0,PLI=2101.0,DT=0.0833333,

---------------------------------------------------------------------
MODEL EQUATIONS
---------------------------------------------------------------------
HOR=P/H
PHM=PHMT(HOR)
PBR=P/B
PJM=PJMT(PBR)
PMM=PMF*PHM*PJM
PM=P*PMN*PMM
PB=PBN*P
PD=P*PDN
HPM=HPMT(HOR)
BGT=(B-BA)/(B*BAT)
HBM=HBMT(BGT)
HM=HF*HPM*HBM
HC=H*HCN*HM
HO=H*HON
BPM=BPMT(PBR)
BGM=BGMT(BGT)
BM=BF*BPM*BGM
BC=B*BCN*BM
BO=B*BON
PLM =HM*PLHF
PLS=PL*PLSN*PLM
---------------------------------------------------------------------
EVALUATION OF STATE VARIABLES
---------------------------------------------------------------------
P=P+DT*(PB+PM-PD)
H=H+DT*(HC-HO)
B=B+DT*(BC-BO)
BA=BA+DT*(B-BA)/BAT
PL=PL+DT*PLS
---------------------------------------------------------------------
LIST OF SYMBOL NAMES
---------------------------------------------------------------------
   P=POPULATION OF THE TOWN                                (PEOPLE)
  PI=INITIAL POPULATION OF THE TOWN                        (PEOPLE)
  PB=SUM OF ANNUAL BIRTHS AND DEATHS                  (PEOPLE/YEAR)
  PM=ANNUAL MIGRATIONS TO THE TOWN                    (PEOPLE/YEAR)
  PD=ANNUAL DEPARTURES FROM TOWN                      (PEOPLE/YEAR)
 PHM=POPULATION HOUSING MULTIPLIER                  (DIMENSIONLESS)
 PBR=POPULATION BUSINESS RATIO                    (PEOPLE/BUSINESS)
 PJM=POPULATION JOBS MULTIPLIER                     (DIMENSIONLESS)
 PMM=POPULATION MIGRATION MULTIPLIER                (DIMENSIONLESS)
 PMF=POPULATION MIGRATION FACTOR                    (DIMENSIONLESS)
 PMN=POPULATION MIGRATION NORMAL                    (FRACTION/YEAR)
 PBN=POPULATION GROWTH NORMAL                       (FRACTION/YEAR)
 PDN=POPULATION DEPARTURE NORMAL                    (FRACTION/YEAR)
   H=NUMBER OF HOUSES IN THE TOWN                          (HOUSES)
  HI=INITIAL NUMBER OF HOUSES IN THE TOWN                  (HOUSES)
  HC=ANNUAL HOUSING CONSTRUCTIONS                     (HOUSES/YEAR)
  HO=ANNUAL AGEING OF HOUSES                          (HOUSES/YEAR)
 HOR=HOUSING OCCUPANCY RATE                          (PEOPLE/HOUSE)
 HPM=HOUSING PUPULATION MULTIPLIER                  (DIMENSIONLESS)
 HBM=HOUSING BUSINCSS MULTIPLIER                    (DIMENSIONLESS)
  HM=HOUSING MULTIPLIER                             (DIMENSIONLESS)
  HF=HOUSING FACTOR                                 (DIMENSIONLESS)
 HCN=NORMAL HOUSING CONSTRUCTION RATE               (FRACTION/YEAR)
 HCN=NORMAL HOUSING AGEING RATE                     (FRACTION/YEAR)
   B=NUMBER OF BUSINESSES IN THE TOWN                  (BUSINESSES)
  BI=INITIAL NUMBER OF BUSINESSES IN THE TOWN          (BUSINESSES)
  BC=ANNUAL BUSINESS CONSTRUCTIONS                (BUSINESSES/YEAR)
  BO=ANNUAL AGEING OF BUSINESSES                  (BUSINESSES/YEAR)
 BGT=BUSINESS GROWTH RATE                           (FRACTION/YEAR)
 BPM=BUSINESS POPULATION MULTIPLIER                 (DIMENSIONLESS)
 BGM=BUSINESS GROWTH MULTIPLIER                     (DIMENSIONLESS)
  BM=BUSINESS MULTIPLIER                            (DIMENSIONLESS)
  BF=BUSINESS FACTOR                                (DIMENSIONLESS)
 BCN=NORMAL BUSINESS CONSTRUCTION RATE              (FRACTION/YEAR)
 BON=NORMAL BUSINESS AGEING RATE                    (FRACTION/YEAR)
  BA=AVERAGE NUMBER OF BUSINESSES                      (BUSINESSES)
 BAI=INITIAL AVERAGE NUMBER OF BUSINESSES              (BUSINESSES)
 BAT=BUSINESS AVERAGING TIME                                (YEARS)
  PL=NUMBER OF PLOTS SOLD                                  (HOUSES)
 PLI=INITIAL NUMBER OF PLOTS SOLD                          (HOUSES)
 PLS=ANNUAL PLOT SALES                                (HOUSES/YEAR)
PLSN=NORMAL SALES RATE OF PLOTS                     (FRACTION/YEAR)
 PLM=PLOTS MULTIPLIER                               (DIMENSIONLESS)
PLHF=PLOTS-HOUSING FACTOR                           (DIMENSIONLESS)
  DT=SOLUTION INTERVAL FOR THE SIMULATION                   (YEARS)
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VALUES OF STATE VARIABLES FOR THE SIMULATION PERIOD JAN. 1982 TO JAN. 1987

TIME	POP.(P)	HOUSES(H)	PLOTS(PL)	BUS.(B)	BUS.AV.(BA)
1982.0000	9571.0000	1764.0000	2101.0000	324.0000	324.0000
1982.0833	9709.1608	1790.8990	2135.5991	324.2764	324.0058
1982.1667	9848.7031	1818.1667	2170.7148	324.6036	324.0182
1982.2500	9989.6488	1845.8056	2206.3517	324.9816	324.0383
1982.3333	10132.0184	1873.8185	2242.5142	325.4105	324.0669
1982.4167	10275.8315	1902.2080	2279.2070	325.8903	324.1049
1982.5000	10421.1067	1930.9770	2316.4351	326.4211	324.1531
1982.5833	10567.8619	1960.1283	2354.2033	327.0029	324.2125
1982.6667	10716.1144	1989.6649	2392.5167	327.6358	324.2838
1982.7500	10865.8807	2019.5899	2431.3802	328.3198	324.3679
1982.8333	11017.1769	2049.9062	2470.7991	329.0550	324.4655
1982.9167	11170.0188	2080.6169	2510.7785	329.8416	324.5775
1983.0000	11324.4216	2111.7252	2551.3239	330.6796	324.7047
1983.0833	11480.4004	2143.2343	2592.4404	331.5690	324.8477
1983.1667	11637.9699	2175.1474	2634.1337	332.5101	325.0073
1983.2500	11797.1447	2207.4677	2676.4091	333.5029	325.1843
1983.3333	11957.9394	2240.1985	2719.2722	334.5475	325.3794
1983.4167	12120.3684	2273.3432	2762.7285	335.6440	325.5932
1983.5000	12284.4460	2306.9050	2806.7838	336.7926	325.8265
1983.5833	12450.1865	2340.8875	2851.4437	337.9935	326.0800
1983.6667	12617.6045	2375.2939	2896.7140	339.2466	326.3543
1983.7500	12786.7144	2410.1277	2942.6004	340.5522	326.6501
1983.8333	12957.5309	2445.3923	2989.1088	341.9105	326.9680
1983.9167	13130.0054	2481.0982	3036.2542	343.3215	327.3087
1984.0000	13303.8767	2517.2794	3084.0826	344.7854	327.6728
1984.0833	13479.1703	2553.9396	3132.6004	346.3022	328.0609
1984.1667	13655.9114	2591.0823	3181.8139	347.8720	328.4736
1984.2500	13834.1251	2628.7114	3231.7295	349.4947	328.9116
1984.3333	14013.8362	2666.8306	3282.3538	351.1706	329.3753
1984.4167	14195.0696	2705.4440	3333.6936	352.8995	329.8654
1984.5000	14377.8499	2744.5553	3385.7555	354.6816	330.3824
1984.5833	14562.2019	2784.1687	3438.5465	356.5170	330.9269
1984.6667	14748.1501	2824.2881	3492.0735	358.4058	331.4993
1984.7500	14935.7193	2864.9179	3546.3437	360.3480	332.1004
1984.8333	15124.9341	2906.0621	3601.3642	362.3438	332.7304
1984.9167	15315.8196	2947.7251	3657.1423	364.3933	333.3901
1985.0000	15508.4005	2989.9112	3713.6854	366.4965	334.0798
1985.0833	15702.7020	3032.6248	3771.0011	368.6537	334.8001
1985.1667	15898.7493	3075.8704	3829.0969	370.8650	335.5514
1985.2500	16096.5680	3119.6525	3887.9805	373.1305	336.3343
1985.3333	16296.2408	3163.9758	3947.6598	375.4488	337.1492
1985.4167	16497.8765	3208.8451	4008.1430	377.8177	337.9965
1985.5000	16701.4973	3254.2655	4069.4385	380.2377	338.8765
1985.5833	16907.1256	3300.2423	4131.5553	382.7088	339.7896
1985.6667	17114.7845	3346.7808	4194.5020	385.2315	340.7363
1985.7500	17324.4971	3393.8862	4258.2876	387.8060	341.7170
1985.8333	17536.2874	3441.5642	4322.9211	390.4326	342.7319
1985.9167	17750.1795	3489.8199	4388.4116	393.1115	343.7814
1986.0000	17966.1978	3538.6591	4454.7684	395.8432	344.8661
1986.0833	18184.3675	3588.0873	4522.0008	398.6279	345.9861
1986.1667	18404.7140	3638.1101	4590.1181	401.4658	347.1419
1986.2500	18627.2631	3688.7332	4659.1301	404.3574	348.3339
1986.3333	18852.0412	3739.9624	4729.0462	407.3030	349.5624
1986.4167	19079.0750	3791.8035	4799.8764	410.3028	350.8278
1986.5000	19308.3917	3844.2625	4871.6305	413.3573	352.1305
1986.5833	19540.0189	3897.3453	4944.3184	416.4667	353.4709
1986.6667	19773.9848	3951.0580	5017.9505	419.6315	354.8492
1986.7500	20010.3180	4005.4067	5092.5368	422.8519	356.2659
1986.8333	20249.0474	4060.3976	5168.0878	426.1284	357.7214
1986.9167	20490.2025	4116.0369	5244.6141	429.4613	359.2160
1987.0000	20733.8132	4172.3310	5322.1262	432.8510	360.7500