ORiON, Vol. 6, No. 1, pp. 3-36 ISSN 0259-191 X

DEGENERACY PROBLEMS IN MATHEMATICAL PROGRAMMING AND DEGENERACY GRAPHS

Tomas Gal,* Department of Quantitative Management
University of South Africa, Pretoria

ABSTRACT

Degeneracy may cause various computing and other complications in any mathematical programming problem of the kind where the constraint set defines a convex polyhedral set (particularly, a polytope). In order to be able to study various seemingly independent degeneracy phenomena from a unifying viewpoint a so-called degeneracy graph (DG for short) is defined, and its properties analyzed. Cycling of the simplex method for LP is analyzed and a method to construct cycling examples of arbitrary size is proposed. The neighbourhood problem is solved by a new approach to determine a minimal N–tree (N for neighbour), and an efficient method to determine all vertices of a convex polytope is described. A new version of the simplex method is indicated that does not need Phase 1, should be faster than commercial codes and automatically contains an anticycling device. For a degenerate optimal solution of an LP–problem, sensitivity analysis as well as shadow price determination and interpretation are tackled by using a special class of DG's, the so-called optimum DG's. The connection between weakly redundant constraints, a degenerate optimal solution of the associated LP and sensitivity analysis as well as shadow price determination is analyzed.

Key words: Linear Programming, Degeneracy, Graph Theory, Degeneracy graphs (Convex Polytopes, Sensitivity analysis, Cycling of the simplex method, Shadow price, Redundancy)

Abbreviated title: DEGENERACY GRAPHS

* On sabbatical leave from Fernuniversität, Hagen, FRG

1. INTRODUCTION

Degeneracy is a complex phenomenon linked to various mathematical programming problems of which the constraint set defines a convex polyhedral set (particularly, a convex polytope), such as linear programming (see, e.g., Altman (1964), Balinski et al. (1986), Beale (1955), Dantzig (1963), Hoffmann (1953), Megiddo (1986), Wolfe (1963)), transportation (network) type of problems (see, e.g., Ahrens und Finke (1975), Cunningham (1979), Cunningham and Klincewicz (1981), McKeown (1978)), quadratic optimization (see, e.g., Chang and Cottle (1980)), linear complementarity problems (see, e.g., Kostreva (1979)), bottleneck linear programming (see, e.g., Derigs (1982), Garfinkel and Rao (1976), Hammer (1969), Seshan and Achary (1982)), multiparametric linear programming (see, e.g., Gal (1979)), linear vector maximization (see, e.g., Gal (1977), (1987), Philip (1977)), linear integer vectormaximum problems (see, e.g., Ramesh et al, (1987)), nonlinear programming (see, e.g. Horst et al. (1988)), piecewise–linear programming (see, e.g., Fourer (1987)) etc. – for more references see Kruse (1986) and Zörnig (1989).

Degeneracy may cause various kinds of computing difficulties, e.g. in vertex searching methods (see, e.g., Altherr (1975), Dyer and Proll (1977, 1982), Mattheiss and Rubin (1980)), integer linear programming (see, e.g., Fleischmann (1970), Nygreen (1987), Young (1968)), simplex methods for linear programming because of cycling (see, e.g., Cameron (1987), Cirina (1985), Dantzig et al. (1955), Hattersley and Wilson (1988), Kotiah and Steinberg (1977, 1978), Magnanti and Orlin (1988), Majthay (1981), Ryan and Osborne (1988), Telgen (1980)), in connection with the determination, definition and interpretation of shadow prices (see, e.g., Akgül (1984), Aucamp and Steinberg (1982), Knolmayer (1976), Mlynarovic (1988), Proll (1987), Strum (1969), Williams (1963) – see also the survey in Gal (1986)), with sensitivity analysis (see, e.g., Evans and Baker (1982), Greenberg (1986), Knolmayer (1984), and the survey in Gal (1986)), in determining neighbouring vertices of a degenerate vertex (see, e.g. Dyer and Proll (1977 and 1982), Mattheis and Rubin (1980) and Kruse (1986)).

Until now each kind of degeneracy problem has been treated separately. Therefore, an attempt has been made to find a unifying approach, a roof under which all the particular degeneracy problems can be subsumed, i.e. analyzed and discussed from a common point of view. This should yield a better understanding of what is behind various problems caused by degeneracy, and when or under which conditions various degeneracy problems arise.

With respect to convex polytopes, a vertex $x^k \in X := \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, is said to be degenerate if it is (geometrically) over–determined (i.e. more than n hyperplanes pass through x^k). To such a vertex a set of bases of the corresponding enlarged matrix $\bar{A} = (A \mid I_m)$ is assigned.

The complex structure of a degenerate vertex or the associated set of bases can be studied by translating such structures into the language of graph theory. An appropriate degeneracy graph (DG for short) is defined, and this yields the instrument suitable for studying and discussing various degeneracy phenomena from a common point of view.

After some preliminary formal remarks in Section 2.1, DG's are introduced; some of their main properties have already been described in Gal et al. (1988). In Section 2.2 a general theory of the DG's for arbitrary degeneracy degree, σ, is indicated. In the subsequent Section 3 to 6 applications are discussed. In Section 3 the neighbourhood problem and the principles of a new solution approach to solve this problem are presented. Section 4 is a concise analysis of why and when cycling of the simplex method occurs. A method is also suggested to construct LP's of arbitrary size that cycle with the simplex method. In Section 5 a degenerate optimal solution of an LP is tackled, and sensitivity analysis, shadow price determination and interpretation are discussed in the light of the so-called optimum degeneracy graphs. In Section 6 some connections between weakly redundant constraints on the one hand, and degeneracy, shadow prices and sensitivity analysis on the other hand are analyzed. Open questions free for further research are mentioned. A very concise survey of the above questions has already been published in Gal et al. (1986) and (1988).

This paper does not give a general survey of degeneracy phenomena, though the references on the topics are quite comprehensive. The intention of this paper is rather to give a survey of the theoretical results of degeneracy graphs and their use for various applications in what we believe to be a completely new approach to tackling degeneracy phenomena.

2. THEORY OF DEGENERACY GRAPHS

2.1 PRELIMINARY REMARKS

Consider the system of inequalities

(2.1) $$Ax \leq b, \; x \geq 0,$$

the corresponding solution set of which is

(2.2) $$X := \{x \in \mathbb{R}^n \,|\, Ax \leq b, \; x \geq 0\},$$

where $A \in \mathbb{R}^{m \times n}$ is a constant matrix, $b \in \mathbb{R}^m$ a constant vector. Without loss of generality we assume throughout that $X \neq \phi$ and bounded, hence X is a convex polytope that has at least one vertex.

Introducing slacks $s = (x_{n+1}, \dots, x_{n+m})^T$, the corresponding solution set is defined by

(2.3) $$\bar{X} := \{y \in \mathbb{R}^{m+n} \,|\, \bar{A}y = b, \; y \geq 0\},$$

where $\bar{A} = (A\,|\,I_m)$, I_m = identity matrix, $y = \binom{x}{s}$.

CASE A. ASSUME THAT THERE IS NO DEGENERACY

To each vertex $x^k \in X$ a feasible basis B_k (regular m x m submatrix of A) is uniquely assigned and vice versa. After appropriately rearranging indices, the tableau defined by

(2.4) $$B_k^{-1}A\,|\,B_k^{-1}\,|\,B_k^{-1}b \xrightarrow{\text{elementwise}} (y_{ij})_{m\,,\,n}\,|\,(\beta_{ij})_{m\,,\,n}\,|\,(\beta_i)_{m\,,\,1}$$

is uniquely assigned to B_k and likewise denoted by B_k. Any basis $B_k = (\bar{a}^{j1}, \dots, \bar{a}^{jm})$, $\bar{a}^{ji} \in \mathbb{R}^m$ columns of $\bar{A}$ for i = 1, ... , m, can be uniquely characterized by a so-called basic-index $\{j_1, \dots, j_m\}$ also denoted by B_k. By this we have established

EQUIVALENCE 1

Vertex $x^k \in X \longleftrightarrow$ Feasible basis $B_k \longleftrightarrow$ Basic-index $B_k \longleftrightarrow$ Tableau B_k in the sense that the terms vertex, basis, tableau and basic-index are interchangeable and synonymous.

The structure of a convex polytope X (see (2.2)) can be characterised using the "Grünbaum Definition" (Grünbaum (1967)) of <u>the graph of the polytope</u>: $G' := G'(X) := (V', E')$, where V' is the set of nodes defined by a one-to-one correspondence between the vertices of X and the nodes of G', while E' is the set of edges defined by a one-to-one correspondence between the edges (1-faces) of X and the edges of G'.

Associating with each vertex $x^k \in X$ the corresponding basis B_k (or: tableau B_k etc. - see equivalence 1), each node B_k of G' can be uniquely associated with a basis B_k. Let x^k, $x^{k'}$ be two neighbouring (adjacent) vertices of X and $(x^k, x^{k'})$ the edge of X connecting x^k with $x^{k'}$. Let B_k and B_k', be the (unique) bases associated with x^k and $x^{k'}$ respectively. Then, moving from x^k to $x^{k'}$(or vice versa) along the edge $(x^k, x^{k'})$ corresponds to a basis exchange from B_k to B_k' (or vice versa). Because $x \geq 0$ in X, a basis exchange from B_k to B_k' corresponds to a positive pivot-step, notation $B_k \overset{+}{\longleftrightarrow} B_k'$. Hence

(2.5) $\quad G' := (V', E')$

where

$$V' := \{B_k \mid k = 1, \dots, K\}, \text{ with K the number of vertices in X and}$$

$$E' := \{\{B_k, B_k'\} \subset V' \mid B_k \overset{+}{\longleftrightarrow} B_k'\}.$$

By this we have established

EQUIVALENCE 2

Vertex $x^k \in X \longleftrightarrow$ feasible basis, tableau, basic index B_k

$\longleftrightarrow$ node B_k of G'.

Corresponding illustrations are to be found in Gal et al. (1988).

CASE B. ASSUME THAT X HAS AT LEAST ONE DEGENERATE VERTEX

Suppose that the vertex $x^0 \in X$ is overdetermined, i.e. more than n hyperplanes pass through $x^0 \in \mathbb{R}^n$. Overdetermined means degenerate (cf. also Hadley (1977), 180 - 181 and Nelson (1957)); in a corresponding simplex type tableau (cf. (2.4)) the value of at least one of the basic variables equals zero. Let us call the number of zero basic variables the <u>degeneracy degree</u>, σ, of the vertex x^0 and the vertex x^0 a <u>σ- degenerate vertex</u>.

If $x^0 \in X$ is a σ-degenerate vertex, then, as is very well known, several bases are associated with x^0. Let

(2.6) $$B^0 := \{B_u^0 | u = 1, \dots, U\}, U > 1,$$

be the set of bases (tableaux) associated with x^0.

2.2 DEGENERACY GRAPHS AND SOME OF THEIR PROPERTIES

In order to be able to study the structure of a degenerate vertex x^0 let us introduce a special graph G_+^0 (x^0) which is defined as follows (Gal (1978 and 1985) and Kruse (1986)):

(2.7) $$G_+^0(x^0) := G_+^0 := (V, E_+)$$

where

$$V := B^0 , E_+ := \{\{B_u^0 , B_{u'}^0\} \subset B^0 | B_u^0 \xleftrightarrow{+} B_{u'}^0\},$$

$$u, u' \in \{1, \dots, U\}, u \neq u'.$$

The graph G_+^0 is called the <u>positive degeneracy graph of x^0</u> (positive DG for short). If x^0 is nondegenerate then $G_+^0 = (B^0,\phi)$ with $B^0 = \{B_0\}$ and $U = 1$.

EQUIVALENCE 3

σ-degenerate vertex $x^0 \in X \longleftrightarrow$ set B^0 of bases $\longleftrightarrow$ Set B^0 of tableaux $\longleftrightarrow$ Set B^0 of basic indices $\longleftrightarrow$ Set B^0 of nodes

Notice that the correspondence is no longer "one-to-one" but "one-to-many".

In the case of degeneracy it is possible to pivot in a tableau with zero basic variables also on negative elements, notation $B_k \overset{-}{\longleftrightarrow} B_k'$, of course "staying" on the same (geometrical) vertex.

The corresponding DG is then called the negative degeneracy graph of x^0 (negative DG for short), denoted by G_-^0.

Using any nonzero pivot, i.e. $B_k \longleftrightarrow B_k'$, the corresponding graph is called the general DG of x^0 denoted by G^0.

The main characteristics for a DG are the degeneracy degree, σ, of the σ-degenerate vertex $x^0 \in X \subseteq \mathbb{R}^n$ and the size of n of $\mathbb{R}^n$. Therefore, a DG is concisely described as a σ x n-DG.

Let us call the representation graph, G(X), (see also Altherr (1975), Jansson (1985), Kruse (1986)) the graph based on G'(X) in which the corresponding positive or negative or general degeneracy graph $G_\pm^i$ is embedded for each degenerate vertex x^i.

Kruse (1986) studied some of the properties of a DG (the corresponding generalizations and proofs are therein):

(1) The positive and the negative DG's may be disconnected; the general DG's are always connected (see also Jansson (1985) and Zörnig (1989)).

(2) If the degeneracy degree is σ and if all neighbouring vertices of the given σ-degenerate vertex x^0 are nondegenerate, then there are $\sigma + 1$ edges in the representation graph connecting any node assigned to a neighbouring vertex of x^0 with some specific nodes in G^0 (or in G_+^0 or in G_-^0).

Kruse (1986) (see also Gal (1978) and (1985)) defines:
The node-set of G^0 (or G^0_+ or G^0_-) that connects G^0 with at least one node of G(X) not in G^0 is called the set of transition nodes. The nodes of G^0 that have no connection to "outer" nodes are called internal nodes.

Another question concerns the number $U = |B^0|$ (cardinality of B^0) of nodes in G^0 (in G^0_+, in G^0_-). The maximum possible number, U_{max}, of nodes in G^0 is obviously

$$(2.9) \qquad U_{max} = \binom{n+\sigma}{\sigma}$$

and it has been proved in Kruse (1986) that the minimal number, U_{min}, of nodes in G^0 is

$$(2.10) \qquad U_{min} = 2^{\sigma-1}(n - \sigma + 2), \qquad \sigma < n.$$

In order to given the reader a feeling of how many nodes a σ x n-DG might have, compare Tab. 2.1 for some selected n and σ.

Tab. 2.1

n	σ	U_{min}	U_{max}
5	3	16	56
10	5	112	3003
50	5	752	$3.48 \cdot 10^6$
50	10	21504	$1.62 \cdot 10^{17}$
50	40	$6.59 \cdot 10^{12}$	$5.99 \cdot 10^{25}$
100	30	$3.865 \cdot 10^{10}$	$2.61 \cdot 10^{39}$
100	50	$2.93 \cdot 10^{16}$	$2.01 \cdot 10^{40}$
100	80	$1.33 \cdot 10^{25}$	3.10^{52}

Consider the σ-degenerate vertex $x^0 \in \mathbb{R}^n$ with the bases set B^0 and let $B^0_u \in B^0$ be a corresponding tableau. Rearranging the indices appropriately tableau B^0_u has the form of Tab. 2.2:

Tab 2.2

B_u^0	$x_1 \ldots x_\sigma$	$x_{\sigma+1} \ldots x_m$	$x_{m+1} \ldots x_{m+n}$	x_B
1	$1 \ldots 0$	$0 \ldots 0$		0
$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$(Y_{\sigma,n})$	$\vdots$
σ	$0 \ldots 1$	$0 \ldots 0$		0
$\sigma + 1$	$0 \ldots 0$	$1 \ldots 0$		$\beta_{\sigma+1} > 0$
$\vdots$	$\vdots \quad \vdots$	$\vdots \quad \vdots$	$(Y_{m-\sigma,n})$	$\vdots$
m	$0 \ldots 0$	$0 \ldots 1$		$\beta_m > 0$

Here $(Y_{\sigma,n})$ and $(Y_{m-\sigma,n})$ are the corresponding parts of the matrix $(B_u^0)^{-1}\bar{A}$ that are associated with the nonbasic variables. The part $(Y_{\sigma,n})$ of the tableau B_u^0 is called the (reduced) σ x n-degeneracy tableau.

In order to be able to derive specific properties of σ x n-DG's it is necessary to find out what kind of graph a DG is. Zörnig (1989) therefore analysed the general structure of DG's for arbitrary σ. For the particular case, $\sigma = 2$, it is possible, however, to use a simpler approach based on the notion of a line graph (see, e.g., Beineke and Wilson (1978) and Zörnig (1989)).

Using this theory various properties of the σ x n-DG's can be derived. Formulas for determining the number of nodes of a σ x n-DG are found (Zörnig (1989)). Other derived properties of the σ x n-DG's are:

(1) It holds for the diameter d of a σ-DG that
$$d \leq \min \{\sigma, n\}.$$
(2) From (1) the assertion which has been proved in Kruse (1986) follows immediately, namely, that general σ x n-DG's are always connected.
(3) The connectivity of a σ x n-DG is ≥ 2.
(4) From (3) it follows: Let B_k, $B_{k'}$, $k \neq k'$, be two nodes of a σ x n-DG. Then there exists a closed line in the DG that includes both B_k and $B_{k'}$.

3. THE NEIGHBOURHOOD PROBLEM

The first part of this section could be entitled "The history of DG's" because the problem defined below is the reason why DG's were introduced and have been dealt with since Gal (1978).

Specifically, in some mathematical programming problems it is required to find all or a specific subset of the neighbouring vertices of a given vertex $x^0 \in X \subset \mathbb{R}^n$ (see Introduction for references).

If x^0 is nondegenerate this task is an easy one. If, however, x^0 is a σ-degenerate vertex it was previously normal to implicitly generate all the bases of the set B^0 to ensure having found all neighbours of x^0 (for the imaginable complexity cf. Table 2.1, Cunningham (1979) for the problem of "stalling", and Megiddo (1986) who shows that exiting a degenerate vertex is as hard as solving a general LP-problem; Balinski et al. (1986) deal for such a case with the length of a directed path through G_+^0 (our notation)).

In the framework of determining all vertices of a convex polytope X, Dyer and Proll (1977 and 1982) proposed a method to avoid such tremendous and perhaps superfluous work.

We call the problem of determining all neighbouring vertices of a σ-degenerate vertex $x^0 \in X$ the <u>neighbourhood problem</u> (N-problem for short). Using the notion of a DG, a new way to solve this problem has been found.

Let us first observe that two cases should be considered in general:

(1) All neighbours of the σ-degenerate vertex x^0 are nondegenerate vertices, and
(2) Some of the neighbours are degenerate themselves.

The more general case (2) is tackled in Kruse (1986); for simplicity we assume here that all neighbours of the σ-degenerate vertex x^0 are nondegenerate vertices.

Let us first recapitulate some necessary basic notions:

Definition 3.1 (Kruse (1986), p. 66):

Let G^0 be the general DG of a σ-degenerate vertex x^0. Then a subgraph $\hat{G}^0$ is called a neighbourhood correspondence (N-correspondence for short) of x^0 or G^0 if each neighbouring vertex x^s of x^0 is assigned to at least one node of $\hat{G}^0$ which corresponds to x^s.

Definition 3.2 (Kruse (1986), p. 67):

A subgraph $\hat{G}^0$ of G^0 satisfies the neighbourhood condition (N-condition for short) if it is an N-correspondence of x^0 and connected.

The main theoretical result can then be summarized as follows:

Theorem 3.1 (Gal (1978 and 1985)):

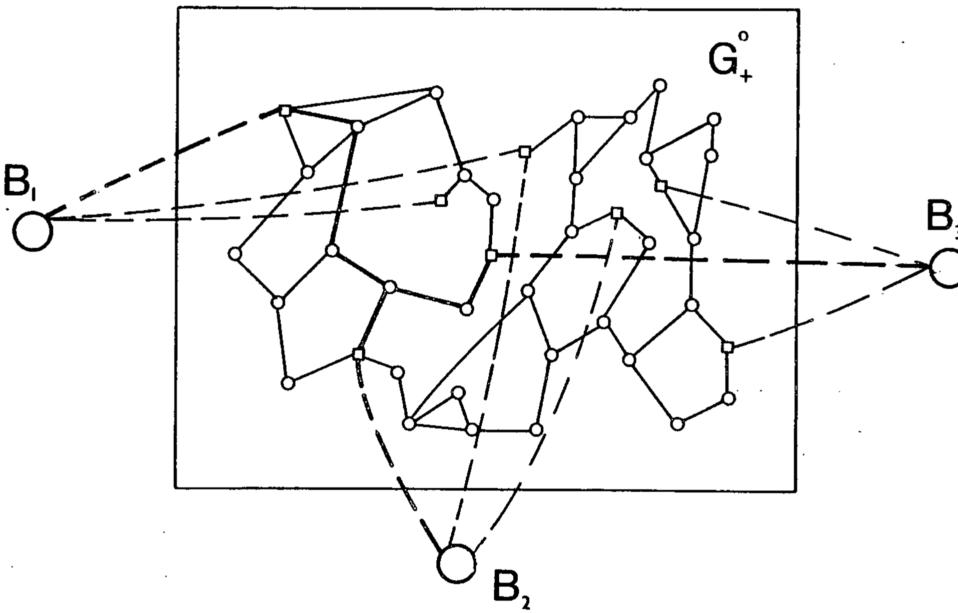
In G^0_+ there exists a tree $\bar{G}^0_+ \subseteq G^0_+$ that

(1) connects G^0_+ with all its adjacent nodes in $G(X)$ not in G^0_+, and
(2) satisfies the N-condition.

A tree $\bar{G}^0_+$ of G^0_+ with the properties in Theorem 3.1 is called an <u>N-tree</u>.

Let us illustrate such a tree in a G^0_+ associated with a 2-degenerate vertex $x^0 \in \mathbb{R}^8$ with the aid of a hypothetical DG in Fig. 3.1; the edges of $\bar{G}^0_+$ are bold lines.

Fig. 3.1



□ - transition nodes

O - internal nodes

..... - edge of a minimal N-tree

——— - edges of an N-tree

In Fig. 3.1 the nodes in the frame are the transition nodes. The nodes B_1, B_2, B_3 of the representation graph are associated with the (non-degenerate) vertices x^1, x^2, x^3, respectively, which are neighbours of x^0.

As is seen from this example and as follows from the proof of Theorem 3.1 several N-trees $\bar{G}^0_+$ may exist.

Further research into this problem resulted in the theoretical basis for a method of determining a minimal N-tree, $\bar{G}^0_{+min}$. It is based on a new pivot selection rule; pivot steps according to this rule are guaranteed to lead from one transition system of the corresponding DG to another. This rule is therefore called the <u>transition node pivoting rule</u> - TNP-rule for short. The principle is as follows: Let x^s be a nondegenerate neighbouring vertex of the σ-degenerate vertex x^0 with the associated tableaux B_s. Pivoting from B_s into a transition node $B^0_u \epsilon B^0$, a nonbasic column t in B^s which is different from the pivot column is selected. After the pivot step is performed, column t in B^0_u has only nonpositive entries in the rows in which the basic variables are zero. Let us refer to such columns (with only nonpositive entries) as "transition columns". Pivoting in B^0 the pivot elements are selected such that column t remains the transition column.

This way of selecting pivots implies a new subgraph, G^0_+ (t) of G^0_+, which is called the <u>t-transition-degeneracy-graph</u>.

It has been proved:

Theorem 3.2 (Geue (1989a))

> Let all neighbours of the σ-degenerate vertex x^0 be nondegenerate. Then the t-transition-degeneracy graph G^0_+ (t) satisfies the N-condition.

Following Definition 3.2 the graph G^0_+ (t) is obviously connected.

Based on Theorem 3.2 the following theorem could be proved:

Theorem 3.3 (Geue (1989a))

In G_+^0 there exists a minimal N-tree, $\bar{G}^0_{+min}$, that connects all transition systems in G_+^0 without using any internal node of G_+^0.

For illustration see Fig. 3.1.

A computerised algorithm is being developed to determine $\bar{G}_+^0$. Theorem 3.3 implies that there cannot be a more efficient procedure to determine all neighbouring vertices of a σ-degenerate vertex x^0 irrespective of the performance of the algorithm. Thus, the N-problem can now be regarded as solved.

The TNP-rule can be used to handle quite unrelated problems. One of them is to determine all vertices of a convex polytope. The basic idea is to transform X (cf. (2.2)) from $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$, by adding b to the columns of A, such that an (m-1)-degenerate vertex $x' \in \mathbb{R}^{n+1}$ of the enlarged system results. A similar idea has been used by Mattheiss (1983) and by Eiselt and Sandblom (1985). It is easy to prove that every vertex of X is a neighbour of x'. Determining $\bar{G}^{\prime}_{+min}$ of $G_+^{\prime}$ (x'), it is an easy task to find all neighbours of x', i.e. all vertices of X. A corresponding computerized algorithm is being developed.

Another consequence of the above theory, which is still under theoretical consideration, is the possibility to modify the simplex method for solving LP's so that an optimal solution can be found in far fewer iterations than in any commercial code (including various crash versions).

4. CYCLING OF THE SIMPLEX METHOD

In any mathematical programming problem in which the constraint set is given by X and for the solution of which some version of the simplex method is used, cycling may occur. Some years ago the opinions on whether or not cycling occurs in real-world problems using professional codes have been divergent (see, e.g., Gass (1979), Kotiah and Steinberg (1977), (1978), Majthay (1981), Telgen (1980)). Regardless of this discussion anticycling methods have evolved since 1952, starting with Charnes' perturbation method (1952). Notice, however, that Dantzig (1963) mentions on pp. 231 f. that A.J. Hoffman constructed the first example of cycling in 1951. In the fall of 1950 Dantzig himself made the first suggestion for an anticycling procedure in a lecture on LP (Gass (1989)).

Subsequently several other anticycling methods have been proposed (see, e.g., Altman (1964), Avis and Chvátal (1978), Azpeitia and Dickinson (1964), Bland (1977), Fleischmann (1970), Harris (1973), Wolfe (1963)).

As recent papers show (cf. Cameron (1987), Cirina (1985), Hattersley and Wilson (1988), Magnanti and Orlin (1988), Ryan and Osborne (1988)), cycling does occur in real-world applications and the cycling problem, especially from the viewpoint of efficient anticycling devices is not yet finally solved. In this connection it is also interesting to mention that one can find only a few different examples for LP's that cycle (Beale (1955), Gassner (1964), Hoffman (1953), Marshall and Suurballe (1969), Cunningham (1979), Solow (1984)).

This can all be regarded as "proof" that the reasons why cycling occurs or under which conditions it occurs are still unknown. Theoretical considerations of the cycling problem can be found in Gassner (1964), Ollmert (1965, 1969), Marshall and Suurballe (1969), who - summarizing concisely - dealt with the question of the minimal number of rows and columns in an LP or in a transportation problem for which cycling can appear at all.

As is seen the cycling problem remains a problem in theory as well as in real-world applications. Regardless of these facts, we consider the question "why or under which conditions does cycling occur" ("why?" for short) as a challenge.

There are several ways to approach the "why"? Let us state firstly that in G_+^0 there are almost always closed lines (circuits for short) which are potentials for simplex cycles (see property of σ x n-DG's in Section 2 and also Fig. 3.1).

The main question on which the research concentrated was: Which properties should a circuit C of G_+^0 have such that C becomes a simplex cycle in the associated LP?

We found that there are several possible concepts that may serve as a basis for answering the question "why?":

(1) The concept of the induced point set
(2) The concept of the induced cone
(3) The concept of the enlarged DG

Because even a concise description of these three concepts would be beyond the scope of this paper we simply refer the reader to Zörnig (1989) for further details.

4.3 CONSTRUCTION OF LP's THAT CYCLE

In order to be able to study the cycling phenomenon in greater depth and from various viewpoints, also with the aid of a computer, it is unavoidable to have as many different LP's that cycle as are needed available. In other words, based on the theory of DG's, a method of constructing cycling examples of arbitrary size m x n had to be created. Based on such examples the following problems can be studied: Does an LP really cycle using commercial codes? Is cycling in a circuit C stable, or - if there are several circuits - does cycling "jump" from one circuit to another (e.g. because of rounding errors), and if yes, what are the consequences for anticycling methods? How good are anticycling devices compared with one another? In other words, which of them use the "shortest path" to get through the set B^0 without cycling (see "stalling" in Cunningham (1979) and Balinski et al. (1986))? How do Bland's pivoting rules (Bland (1977)) behave from the viewpoint of a DG? And so forth.

A general method of constructing cycling LP's of arbitrary size has been found. It should, however, be noted that this general method is not immediately suitable for algorithmic purposes. However, the method indicates the construction of cycling examples of arbitrary size by a successive step-by-step procedure; it uses a system of (determinant-) inequalities (for further details see Zörnig (1989)).

The system of the (determinant-) inequalities can be used for construction of cycling examples starting with a known one. This means that using this idea it is possible to modify or enlarge any given cycling example by one column or one row in one step by solving the corresponding linear determinant-inequalities. Using this procedure which is implemented on an IBM 4341 it is possible to construct cycling examples of arbitrary size (Geue (1989b)).

Incidentally, it can be shown that cycling is prevented when using the TNP-rule.

5. DEGENERACY IN AN OPTIMAL SOLUTION TO AN LP

5.1 OPTIMUM DEGENERACY GRAPHS

Consider

$$\text{(5.1)} \qquad \max_{x \in X} z = c^T x, \qquad c, x \in \mathbb{R}^n$$

with X as in (2.2) and suppose that the σ-degenerate vertex $x^0 \in X$ is an optimal solution of (5.1). According to the definition of the nodes set B^0 of the corresponding (positive, negative or general) DG, the nodes $B_u^0 \in B^0$, $u = 1, \dots, U$, are primal feasible bases of (5.1). As is known, not every primal feasible basis of (5.1) is a dual feasible basis, i.e. optimal basis; this depends on the cost vector c. Hence let $\bar{B}(c)$ be the set of all optimal (i.e. primal and dual feasible) bases of (5.1) and let

$$\text{(5.2)} \qquad \bar{B}^0(c) = B^0 \cap \bar{B}(c)$$

be the set of optimal bases (nodes) of x^0. Assume throughout that $\bar{B}^0(c) \neq \phi$, i.e. x^0 is an optimal vertex. The subgraph $\bar{G}_+^0(c)$ of G_+^0 induced by $\bar{B}^0(c)$ is then called the positive optimum DG (o-DG for short, Kruse (1987)). Similarly for the negative and general o-DG. If c is fixed we simply write $\bar{G}_+^0$, $\bar{G}_-^0$ or G^0, and similarly $\bar{B}^0$ for the node-set.

The theoretical research on o-DG's concentrated on their various properties. The most important preliminary results of this recently started research are summarized in the following lemma.

Lemma 5.1 (Kruse 1987)

(i) The case that $\bar{B}^0(c) = B^0$ exists (i.e. all the bases associated with x^0 are optimal bases)

(ii) There is a triple (A, b, x^0) to which no objective function $z = c^T x$ can be assigned such that $\bar{B}^0(c) = B^0$.

(iii) The case exists that one and only one basis of x^0 is an optimal basis (i.e. $\bar{B}^0(c)$ is a one-element set)

(iv) For the case that one and only one basis of x^0 is an optimal basis it is necessary that $\bar{G}^0_-(c)$ has at least one isolated node.

(v) $\bar{G}^0_-(c)$ can be disconnected.

5.2 SENSITIVITY ANALYSIS UNDER (PRIMAL) DEGENERACY

Sensitivity analysis (SeA for short) with respect to the right hand side b (RHS for short) in (5.1) ("RHS-ranging") or with respect to the objective function coefficients c_j of c in (5.1) ("COST-ranging") has become a constituent part of commercial LP-software. Such rangings are computed in the sense of postoptimal SeA, i.e. first an optimal solution is found, and then SeA is performed.

Let us first represent the simplex tableau of the nondegenerate optimal solution associated with the basis B (Tab. 5.1):

Tab. 5.1

$B^{-1}A$	B^{-1}	$B^{-1}b$		$(y_{ij})_{m,n}$	$(\beta_{ij})_{m,n}$	$(\beta_i)_{m,1}$
$c_B^T B^{-1}A - c^T$	$c_B^T B^{-1}$	$c_B^T B^{-1}b$	$\xrightarrow[\text{wise}]{\text{element-}}$	$(\Delta Z_j)_{1,n}$	$(y_i)_{1,m}$	$\mid z_{max}$

First consider RHS-ranging in terms of

(5.3) $b_i(\lambda) = b_i + \lambda,\ \ i \in \{1, \dots, m\}$ fixed

Since we consider RHS-ranging as a postoptimal analysis, the parameter λ is set to $\lambda = 0$ until an optimal solution to (5.1) is found. Then the LP (5.1) can be rewritten as

(5.4) $\max\limits_{x \in X(\lambda)} z = c^T x$

where

(5.5) $X(\lambda) := \{x \in \mathbb{R}^n \mid Ax \leq b(\lambda), x \geq 0\}$

$b(\lambda) := b + e^i\lambda,$ $\qquad e^i \in \mathbb{R}^m$ unit vector

$i \in \{1, \dots, m\}$ fixed,

and we let $x^0(\lambda)$ be an optimal vertex. According to the requirements of postoptimal analysis we have

$X(0) := X,$

$x^0(0) := x^0.$

Assume first that x^0 is the nondegenerate unique optimal vertex of (5.4). Then <u>the</u> basis B_0 is uniquely assigned to x^0. SeA in terms of (5.3) means the determination of the critical region (interval) Λ^0 such that the basis B_0 remains optimal for all $\lambda \in \Lambda^0$ (see, e.g., Gal (1979)).

Suppose now that x^0 is a σ-degenerate optimal vertex of (5.4) for $\lambda = 0$. The set B^0 is assigned to x^0. Setting $\lambda \neq 0$ and $\lambda \in (-\varepsilon, \varepsilon)$, $\epsilon > 0$ sufficiently small, $\bar{B}^0 \neq \bar{B}^0(\lambda)$ holds in general.

The question hence arises what SeA means in terms of (5.3) in this case. To say "Determine Λ^0, such that for all $\lambda \in \Lambda^0$ the optimal basis (which one?) remains optimal" is dubious.

In papers dealing specifically with such questions (see, e.g., Evans and Baker (1982), Knolmayer (1984)) the proposition is found (using our notation): Determine Λ_k^0 for each $\bar{B}_k^0$, $\quad k = 1, \dots, K,$ $K := |\bar{B}^0|$. Then the <u>overall critical interval</u>, Λ^0, is defined by

(5.6) $\Lambda^0 := \bigcup_{k=1}^{K} \Lambda_k^0.$

This proposition concerns the <u>procedure</u> of determining the overall critical interval Λ^0. However, it does not specify what Λ^0 means, i.e. what should remain invariant for all $\lambda \in \Lambda^0$.

Let $\bar{B}(\lambda)$ be the set of all nodes (optimal bases) of (5.4) and let $\bar{B}^0(\lambda) := \bar{B}(\lambda) \cap \bar{B}^0$ be the set of optimal bases associated with $x^0(\lambda)$; then $\bar{B}^0(0) := \bar{B}^0$.

In Piehler (1988) the following theorem is proved:

Theorem 5.1 (Piehler (1988))

$$\bar{B}^0(\lambda) \neq \phi \Longleftrightarrow \lambda \in \bigcup_{k=1}^{K} \Lambda_k^0$$

SeA in this sense then means to determine the overall critical interval Λ^0 such that for all $\lambda \in \Lambda^0$ at least one optimal basis $\bar{B}_u^0 \in \bar{B}^0$ remains optimal.

Up to this point the problem seems to be solved. Notice, however, that in determining Λ^0 via $\bigcup_k \Lambda_k^0$, dual simplex steps are used (cf. Knolmayer (1984)), i.e. one proceeds along the edges of G_-^0. As Lemma 5.1(v) shows, G_-^0 need not be connected. Which k's in $\bigcup_k \Lambda_k^0$ should then be taken into account? The k's that belong to a component (maximal connected subgraph) of $\bar{G}_-^0$? And if there are several components in $\bar{G}_-^0$, which one should be considered? Or is it necessary to consider all of them? And if there is an isolated node in $\bar{G}_-^0$, is it sufficient to consider this one?

Using Knolmayer's (1984) approach to determine Λ^0 by negative pivots (i.e. going along the edges of the negative o-DG) this algorithm may fail in the case of <u>disconnected</u> negative o-DG's. Therefore, current research concentrates on theoretical backgrounds and on finding a method of determining Λ^0 with minimal effort such that the "true" Λ^0 results.

In this connection notice that if x^0 is an optimal σ-degenerate vertex, then the commercial RHS-codes yield false results.

The problem is analogous in the case of SeA with respect to c_j, $j \in \{1, \dots, n\}$ fixed, in terms of

$$(5.7) \quad c_j(t) = c_j + t, \qquad j \in \{1, \dots, n\} \text{ fixed.}$$

Our proposition for the meaning of COST-ranging in a degenerate case reads:

Let T_k^0 be the critical interval of t in terms of (5.7) associated with the basis B_k^0 and let

$$(5.8) \quad T^0 := \bigcup_{k=1}^{K} T_k^0$$

$$K := |\bar{B}^0|$$

be the overall critical interval for COST-ranging with respect to an optimal σ-degenerate vertex x^0.

Then SeA with respect to c_j means to determine T^0 such that for all $t \in T^0$ at least one $\bar{B}_k^0$ remains optimal or the vertex x^0 remains optimal.

With a few exceptions the open questions in this case are similar to those in the case of RHS-ranging; therefore, we shall not deal with them here. Let us, however, stress that in this case too the commercial COST-ranging codes yield false results.

5.3 SHADOW PRICES IN LP UNDER DEGENERACY

As is very well known, shadow prices are important indicators not only in LP (see, e.g., Proll (1987)). To determine shadow prices in LP is no problem, provided that there is no degeneracy; they can be found immediately in the optimal simplex tableau (in whatever form it is available).

Assume that x^0 is the nondegenerate optimal vertex associated with basis B_0. Let us recall that the i^{th} shadow price, y_i, (cf. also Tab. (5.1)) in LP is defined as

$$(5.9) \quad y_i := \frac{\partial z}{\partial b_i},$$

where y_i is the i^{th} component of $c_{B_0}^T B_0^{-1}$. In the case where x^0 is an optimal σ-degenerate vertex, "two-sided shadow prices" are defined in the literature (Akgül (1984), Aucamp and Steinberg (1982), Strum (1969), Williams (1963) - see also the survey in Gal (1986)) - as follows:

$$(5.10) \quad y_i^+ := \frac{\partial z}{\partial b_i^+},$$

the right side partial derivative

$$y_i^- := \frac{\partial z}{\partial b_i^-},$$

the left side partial derivative.

CASE A

Let x^0 be a nondegenerate optimal vertex. Then

$$(5.11) \quad y_i^+ = y_i^- = y_i \text{ for all } i = 1, \dots, m.$$

CASE B

Let x^0 be an optimal σ-degenerate vertex. Then

$$(5.12) \quad y_i^- \geq y_i^+$$

and, in general

$$y_i^- \neq y_i^+.$$

In the above references y_i^+ is interpreted as "the maximal buying price for one unit of resource b_i", and y_i^- as "the least selling price for one unit of resource b_i".

To determine the two-sided shadow prices, i.e. y_i^+ and y_i^- for all i = 1, ... , m, the following suggestion can be found in Akgül (1984) and Aucamp and Steinberg (1982):

Let $y_i^{(k)}$ be the shadow price associated with tableau $\bar{B}_k^0$ (in our notation). Then

(5.13) $\quad y_i^- = \max\limits_k \{y_i^{(k)}\}, \qquad y_i^+ = \min\limits_k \{y_i^{(k)}\}$

The determination of the two-sided shadow prices is closely related to SeA with respect to RHS. A corresponding parametric approach to determine the two-sided shadow prices has been suggested in Gal (1986). Therefore, the open problems in connection with the determination of shadow prices under degeneracy are analogous to those in SeA with respect to RHS.

5.4 REDUNDANCY, DEGENERACY AND SENSITIVITY ANALYSIS

In the search for reasons for degeneracy, one may say that, in general, the occurrence or non-occurence of degeneracy depends on the structure of the matrix (A, b) (cf. also Greenberg (1986)).

A sufficient condition for degeneracy is weak redundancy (see also Karwan et al. (1983)). Obviously, if a weakly redundant constraint passes through a vertex $x \in X$ it always causes an overdetermination of x and hence degeneracy.

In this section we shall concisely discuss interrelations between degeneracy caused by weak redundancy, SeA and shadow prices. To illustrate the problem, consider

$$\max z = x_1 + x_2$$

$$\begin{aligned} \text{s.t.} \quad x_1 + 6x_2 &\le 30 \\ 2x_1 + x_2 &\le 16 \\ x_1 + 2x_2 &\le 14 \\ 7x_1 + 4x_2 &\le 58 \\ x_1 \ge 0,\ x_2 &\ge 0 \end{aligned}$$

The following four tableaux are the optimal tableaux associated with the optimal 2-degenerate vertex $x^0 = (6, 4)^T$

$\bar{B}^0_1$	3	4	x_B
1	-.09	.54	6
2	.18	-.09	4
5	-.27	-.36	0
6	-.09	-3.45	0
Δz^1_j	.09	.45	10

$\bar{B}^0_2$	3	6	x_B
1	-.1053	.1579	6
2	.1842	-.0263	4
4	.0263	-.2895	0
5	-.2632	-.1053	0
Δz^2_j	.0789	.1316	10

$\bar{B}^0_3$	4	5	x_B
1	.6	-.3	6
2	.3	.6	4
3	1.3	-3.6	0
6	-3.3	-.3	0
Δz^3_j	.3	.45	10

$\bar{B}^0_4$	5	6	x_B
1	-.4	.2	6
2	.7	-.1	4
3	-3.8	.4	0
4	.1	-.3	0
Δz^4_j	.3	.1	10

The set of $\bar{B}^0 \neq B^0$ because there are two more basic indices associated with x^0 (regardless of dual feasibility), i.e. $B^0_5 = \{1, 2, 3, 5\}$ and $B^0_6 = \{1, 2, 4, 6\}$. The graphs G^0_+ and G^0_- associated with x^0 are depicted in Fig. 5.5a, b.

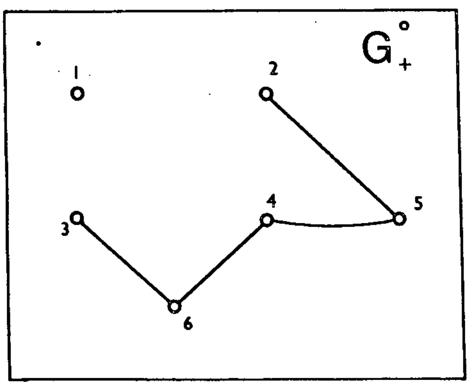


Fig. 5.5a

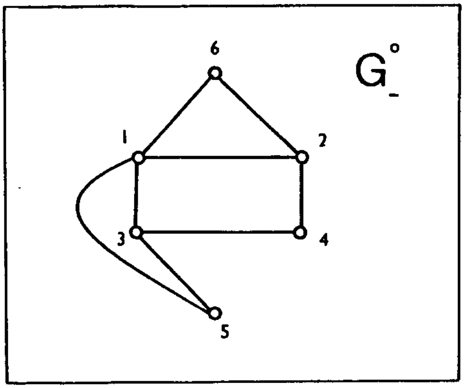


Fig. 5.5b

In Fig. 5.6a and 5.6b the o-DG's $\bar{G}^0_+$ and $\bar{G}^0_-$, are shown respectively.

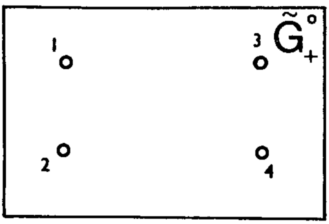


Fig. 5.6a

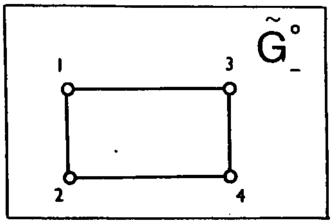


Fig. 5.6b

Tableau $\bar{B}^0_1$ shows that constraints nos 3 and 4 are weakly redundant (the elements associated with the corresponding rows in the nonbasic columns are negative and the value of the corresponding basic variable is zero) - see also Gal (1983).

Using all four optimal tableaux the overall critical interval for λ_1 is

$\Lambda^0 = \bigcup_{k=1}^{4} \Lambda_k^0 = [-22, \infty)$, and for λ_2 it s $[-11, \infty)$, "the" shadow prices considering only tableau B_1^0 would be $y_1 = .09$, $y_2 = .45$.

It is known that omitting (weakly or strongly) redundant constraints does not influence the set X. Hence, if we omitted the weakly redundant constraints in our example (or in general) would we obtain the correct results for SeA with respect to RHS or for the shadow prices from the reduced tableau?

Let us omit the last two rows in Tableau $\bar{B}_1^0$. Then we obtain $\lambda_1 \in [-22, 66]$ and $\lambda_2 \in [-11, 44]$ which is not the same result as above.

One might defend the viewpoint that by omitting the weakly redundant constraints a "cleaned" X is obtained and the optimal vertex remains the same (though nondegenerate). So why deal with such questions? It is known that it does matter whether (weakly or strongly) redundant constraints are omitted or not (see, e.g., Zimmermann and Gal (1975)) from the economical point of view. All papers dealing with SeA or shadow prices under degeneracy (see the survey in Gal (1986)) use illustration examples in which degeneracy is caused exclusively by weak redundancy. From a purely formal point of view one might say: Methods for determining redundant constraints (see Karwan et al (1983)) are used to reduce the number m of rows of A in $Ax \leq b$ by omitting redundancies to save CPU-time for finding an optimal solution of the associated LP. If degeneracy is caused exclusively by weak redundancy, SeA or shadow price determination would be given by the final (optimal, nondegenerate) tableau after finding an optimal solution without redundancies (= no degeneracy).

From the economical point of view this is simply not true. Using all tableaux $\bar{B}_k^0$, k = 1, ... , 4, in our example we find $y_1^+ = 0$, $y_1^- = .09$, $y_2^+ = 0$, $y_2^- = .45$, $y_3^+ = 0$, $y_3^- = .3$, $y_4^+ = 0$, $y_4^- = .1316$. Which are the "true" shadow prices and which are the "true" critical intervals for λ_1 and λ_2?

To answer these questions in all details an additional theoretical formal and economical analysis is needed.

The situation is slightly different in the case of SeA with respect to c. In our example

$$T^0 := \bigcup_{k=1}^{K} T_k^0$$

yields

$t_1 \in [-.83; 1]$, $t_2 \in [-.5; 5]$

Considering Tableau $\bar{B}_1^0$ with and without the last two rows we obtain the same results. The reason for this is obvious: By changing c we change the gradient of $c^T x$. The limits for changing c are reached when c(t) becomes linearly dependent with the gradient of a bounding hyperplane of X that is "incident" on the optimal vertex x. Such a "rotation" is obviously independent of the weakly redundant constraints passing through x. However, when the weakly redundant constraints are included, the "jumps" from T_1^0 to T_2^0 etc. certainly do have some economical interpretation.

6. CONCLUSIONS

Degeneracy may appear in any mathematical programming problem of which the constraints set defines a convex polytope. When degeneracy does occur it may influence the effort required to compute an optimal solution, or it may completely change the determination and interpretation of e.g. sensitivity analysis (SeA), shadow prices etc.
Until now the problems associated with degeneracy have been tackled separately. In order to be able to handle all aspects from a common point of view, so called degeneracy graphs (DG for short) have been defined. A theory of DG's has been developed in which - among other things - the cardinality of the node set, the connectivity, and a general concept have been found. This theory is then used to explain reasons for cycling of the simplex method and to determine all neighbouring vertices of a degenerate vertex with minimal effort (neighbourhood problem). For a degenerate optimal solution of a linear programming problem the theory of so-called optimum DG's has been developed which can be used for explaining why commercial codes fail to give correct results for sensitivity analysis with respect to the RHS or to the cost coefficients and how to overcome difficulties. Finally, the interrelations between weakly redundant constraints, degeneracy, SeA and shadow price determination have been indicated using optimum DG's.

Since the theory of DG's and the application of this theory to various degeneracy phenomena is quite new, there remain several questions yet unsolved. For instance, application of DG's to explain cycling of the simplex method and theoretical considerations of properties of special induced point-sets have not yet been analysed in all ramifications. In this regard methods for constructing examples of cycling LP's with arbitrary size also need to be improved. The theory of optimum DG's is at its very beginning. Therefore, open problems concerning SeA and shadow prices under degeneracy are still not fully solved.

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