OPTIMAL DESIGN OF NETWORK DISTRIBUTION SYSTEMS

URY PASSY[1]
Department of Quantitative Management
UNISA

Abstract

The problem of finding the optimal distribution of pressure drop over a network is solved via an unconstrained gradient type algorithm. The developed algorithm is computationally attractive. Problems with several hundred variables and constraints were solved.

1 Introduction

The optimization problem solved here belongs to a class of engineering optimization problems known as *pipe network optimization*. Such problems are formulated for natural gas, district heating and water supply networks. The constraints imposed on such problems usually represent the demand by the customers, be it natural gas, heat or water. During the last two decades, engineers have developed different algorithms for solving the pipe network optimization problem, Cohen [4], Fujiwara et al. [8], Kally [10], Quindry et al. [16], Shamir [18]. An extensive list of references concerning both the mathematical formulations and algorithms for solving such networks is given in Walski [19]; Alperovitz and Shamir [1] used successive

---

[1]Permanent Address: Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa 32000, Israel

linear programming where the lengths of the pipe segments of a given diameter were the variables; Epp and Fowler [7] described a Newton type method, while Wood and Charles [20] described a linear theory method. Nielson [12] analyzed the existing methods and proposed a robust strategy for switching between the various methods while solving the problem. Different assumptions on the network were done by different people: Hansen [9] assumed that the network did not have any pumps and its layout is given thus—only the pipes' diameters are unknown; Rothfarb [17] and Murtagh [11] assumed that the network is defined by a tree, *i.e.*, no loops. We will assume that the network does not include pumping stations, reservoirs, special purpose valves or loops. Such networks, for example, describe an offshore natural-gas pipeline system, Rothfarb [17]. In our case the network is defined by a directed tree $T = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of edges (see Fig. 1).
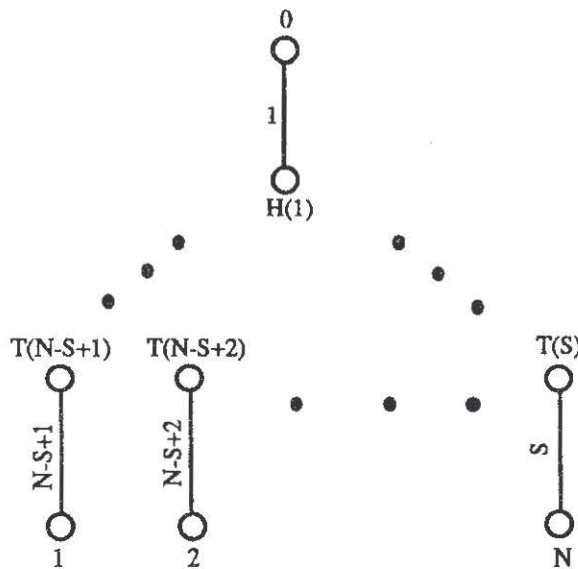

Figure 1. Schematic representation of the network T. It contains S pipe sections. Only the terminal sections are shown in the diagram. $H(A)$ and $T(A)$ denote, respectively, the *Head* and *Tail* of arc A.

It has $N + 1$ terminal nodes, *i.e.*, a single supply node denoted by 0, and N demand nodes denoted consecutively by $1, \ldots, N$. A simple chain $\mathcal{C}_k$ $k = 1, \ldots, N$ leads from the supply node 0 to each demand node. Physically, an arc (i, j) is a tube of specified length, L_{ij}, and diameter, d_{ij}.

Given:

(i) the network configuration

(ii) the pressure P_k $k = 0, \ldots, N$ at the supply node and at each demand node, and

(iii) the flow rate at each of those nodes W_k $k = 1, \ldots, N$,

the determination of the physical dimensions of each arc (i, j) are required so that the total weight (ℓb) of the network is minimized. The weight per unit length of pipes, if the diameter is much larger than the thickness, is a linear function of the diameter, Murtagh [11]. Mathematically, the optimization problem Q is defined by:

$$Q : \min \sum_{(i,j) \in \mathcal{A}} L_{ij} d_{ij} \qquad (1.1)$$

subject to the following constraints:

$$P_i^2 - P_j^2 = k_{ij} L_{ij} / d_{ij}^a \quad (i, j) \in \mathcal{C}_k \quad k = 1, \ldots, N \qquad (1.2)$$

$$\sum_{(i,j) \in \mathcal{C}_k} L_{ij} \geq \mathbf{L}_k \qquad k = 1, \ldots, N \qquad (1.3)$$

where

k_{ij} is a constant which depends on the physical properties of the fluid: viscosity, density, molecular weight, etc. and the flow rate.

$$k_{ij} = \frac{64 f L_{ij} W_{ij}^2 R T C_{ij}}{\pi^2 g M d_{ij}^{0.186}}$$

$$
\begin{aligned}
\text{where}: \quad f \quad &= \quad \text{friction factor} \\
&= \quad 0.0475 \, (Re)^{-0.186} \\
&= \quad 0.0475 \left(\frac{1894 W_{ij}}{\mu d_{ij}} \right)^{-0.186} \\
d_{ij} \quad &= \quad \text{diameter (ft.)} \\
L_{ij} \quad &= \quad \text{length of pipe (ft.)} \\
W_{ij} \quad &= \quad \text{weight rate of flow } (lb/sec) \\
R \quad &= \quad \text{gas constant} = 1546 \, (ft - lb_F / lb \text{ mole } \,^\circ R) \\
M \quad &= \quad \text{molecular weight } (lb/lb \text{ mole}) \\
g \quad &= \quad 32.7 \, (lbft/lb_F \, sec^2.) \\
\mu \quad &= \quad \text{viscosity (centipoise)} \\
C_{ij} \quad &= \quad \text{kinetic energy correction factor} \\
Re \quad &= \quad \frac{1894 W_{ij}}{\mu d_{ij}} \text{ Reynold's number}
\end{aligned}
$$

a is a positive constant which depends on the flow rate. Here the value of a was set equal to 4.814, American Petroleum Institute [2].

$\mathbf{L}_k$ is the distance from the source 0 to demand node k.

By adding Equation (1.2) along each chain, one obtains:

$$\sum_{(i,j) \in \mathcal{C}_k} k_{ij} L_{ij} / d_{ij}^a = P_k^2 - P_0^2 \equiv b_k \quad k = 1, \ldots, N . \qquad (1.4)$$

It can be shown that the constrained set defined by Equations (1.2) and (1.3) is equivalent to that defined by Equations (1.4) and (1.3). Since constraint Equation (1.3) has a *geq* sign, the optimization is a *reversed geometric program*, Duffin and Peterson [6], which is highly nonconvex and can be solved by a branch and bound method, Passy [15]. In that method, a series of *regular geometric* subprograms, Duffin et al. [5],

$$Q(\ell) : \min \sum_{(i,j) \in \mathcal{A}} L_{ij} d_{ij} \, ,$$

subject to Equation (1.4) and to

$$L_{ij} \geq \ell_{ij} \quad \ell_{i,j} \in \ell \, , \tag{1.5}$$

where $\ell = \{\ell_{ij}\}$ is a given set of tube lengths generated by the branching process, are solved at each iteration. It can be shown that if (L_{ij}^*, d_{ij}^*) solves $Q(\ell)$, then $L_{ij}^* = \ell_{ij}$. Hence, during the execution of the above-mentioned branch and bound algorithm, or as in many models *c.f.* Hansen [9], Murtagh [11], where the tubes' lengths are given, the only unknown variables are the diameters, d_{ij}'s, which can be determined after substituting $L_{ij}^* = \ell_{ij}$ in the objective function, and the constraints of $Q(\ell)$

$$Q(\ell) : \quad \min \sum_{(i,j) \in \mathcal{A}} \ell_{ij} d_{ij} \ : \ \sum_{(i,j) \in \mathcal{C}_k} k_{ij} \ell_{ij} / d_{ij}^a \leq b_k \, , \quad k = 1, \ldots, N \, .$$

Since the constraints are each a monotonic decreasing function of the tube's diameters, it was possible to replace the equality constraints (1.4) with inequalities (see Lemma 1).

In the present paper, a fast and simple algorithm for solving $Q(\ell)$ is given, together with convergence properties and results on the rate of convergence. Due to the simplicity and speed, the algorithm can be adapted as a subroutine to networks with loops, reservoirs, etc.. The simplicity of the algorithm stems from the fact that the dual is an unconstrained program. This is an example of a constrained program whose dual is unconstrained, Ben-Tal and Barzilai [3]. The algorithm was tested on problems with several hundred variables and up to one hundred constraints (Eq. (1.4)).

2 Properties of $Q(\ell)$ and The Algorithm

Define the following variables:

$$
\begin{aligned}
x_{ij} &= d_{ij}^{-a} \\
\frac{k_{ij} \ell_{ij}}{b_k} &= \alpha_{ij} \\
|\mathcal{A}| &= S \quad \text{The number of arcs in } \mathcal{A} \text{ which is} \\
&\qquad \text{equal to the number of tube sections .}
\end{aligned}
$$

If this notation is used, then $Q(\ell)$ is defined by:

$$Q(\ell) \; : \quad \min \sum_{(i,j)\in\mathcal{C}} \ell_{ij} x_{ij}^{-\beta} \quad \beta = 1/a \quad (1/a = 0.2077)$$

subject to N linear inequality constraints:

$$\sum_{(i,j)\in\mathcal{C}_k} \alpha_{ij} x_{ij} \le 1 \qquad k = 1,\dots,N$$

$$x_{ij} > 0 \qquad (i,j) \in \mathcal{A} \; .$$

$Q(\ell)$ is a very special *geometric program*. Its objective function is convex and separable, and the constraints are linear.

There are many ways to enumerate the arcs $A \in T$. One way is to enumerate the arcs so that A_1 is the directed arc whose tail, $T(A_1)$, is node 0 and A_{S-N+i} is the directed arc whose head, $H(A_{S-N+i})$ is node i, $i = 1,\dots,N$. With such an enumeration, it is possible to rearrange the variables x_{ij} and the coefficients α_{ij} and ℓ_{ij} in a one-dimensional array. Then:

$$Q(\ell) \; : \quad \min \sum_{j=1}^{S} \ell_j x_j^{-\beta} \triangleq g_0(x)$$

subject to $g(x) \triangleq \mathbf{A}x \le e$, $e = [1,1,\dots,1]^T \in E^N$. Accordingly, $\mathbf{A}$, the matrix of coefficients, has the following form:

$$\mathbf{A} = \left.\begin{bmatrix} a_{11}\; a_{12}\dots & a_{1(S-N+1)}\; 0\dots 0 \\ a_{21}\; a_{22}\dots & \ddots 0 \\ a_{N1}\; a_{N2}\dots & 0\;\dots\; a_{NS} \end{bmatrix}\right\} N, \tag{2.1}$$

$$\underbrace{\phantom{a_{11}\;a_{12}\dots}}_{S-N} \quad \underbrace{\phantom{a_{1(S-N+1)}\;0\dots 0}}_{N}$$

where $\mathbf{A}$ is a non-negative matrix, the first column of $\mathbf{A}$, a^1, is a positive vector and the last N columns have only a single non-zero but positive element, $a_{i(S-N+i)} > 0$. Each row of $\mathbf{A}$ has at most $S - N + 1$ positive elements. Thus, the matrix $\mathbf{A}$ is a highly sparse matrix.

The solution point x^* of $Q(\ell)$ is not changed if, instead of minimizing $g_0(x)$, one minimizes $Cg_0(x)$ where C is a positive constant given by

$$C^{-1} = \beta S^{(\beta+1)} (\max_j \ell_j)(\max_{i,j} a_{ij})^\beta \; . \tag{2.2}$$

Without ambiguity it will be assumed that

$$g_0(x) = C\left(\sum_{j=1}^{S} \ell_j x_j^{-\beta}\right) \triangleq \sum_{j=1}^{S} C_j x_j^{-\beta} \quad (C_j = \ell_j C) \; .$$

This *normalization* is done for the purpose of bounding the Lagrange multipliers from above (see Lemma 7).

Recall that the equality constraints (1.4) were replaced with inequalities . The following lemma justifies this change.

Lemma 1 *At the solution point* $x^*, \mathbf{A}x^* = e$.

Proof. Assume that $g_i(x^*) = a_i^T x^* < 1$. The variable x_{S-N+i} appears only in the i-th constraint. By slightly increasing this component, the objective function is decreased and the new point is still feasible, contradicting the optimality of x^*.

$$\text{Q.E.D.} \quad \square$$

In the following simple lemmas we will show that both $Q(C\ell)$ and its dual have optimal solutions.

Lemma 2 *Program $Q(C\ell)$ is superconsistent [4], i.e., there exists a point x^* such that $\mathbf{A}x^* < e$ and $x^* > 0$.*

Proof. Choose:

$$x_j^* = \frac{1}{(S-N+1)}(\max_i\{a_{ij}\} + 1)^{-1} \quad j = 1, \ldots, S.$$

Then $\mathbf{A}x^* < e$, and $x^* > 0$. Q.E.D. $\square$

The Geometric Dual (GD)$(C\ell)$ of Q$(C\ell)$ is:

$$\text{GD}(C\ell) \ : \ \max V(\delta) \equiv \prod_{j=1}^{S}(C_j/\delta_{0j})^{\delta_{0j}} \prod_{i=1}^{N} \prod_{j \in q(i)} \left(\frac{a_{ij}\gamma_i}{\delta_{ij}}\right)^{\delta_{ij}} \tag{2.3}$$

subject to: **(a)** the normality condition:

$$\sum_{j=1}^{S} \delta_{0j} = 1 \ , \tag{2.4}$$

(b) the orthogonality conditions:

$$\sum_{i \in p(j)} \delta_{ij} = \beta\delta_{0j} \quad j = 1, \ldots, S \tag{2.5}$$

and **(c)** the positivity conditions:

$$
\begin{aligned}
\delta_{ij} &\geq 0 & j &= 1, \ldots, N \ \ j \in q(i) \\
\delta_{0j} &\geq 0 & j &= 1, \ldots, S \\
\gamma_i = \sum_{j \in q(i)} \delta_{ij} &\geq 0 & i &= 1, \ldots, N
\end{aligned}
\tag{2.6}
$$

where

$$
\begin{aligned}
q(i) &= \{j \mid a_{ij} \neq 0\} \\
p(j) &= \{i \mid a_{ij} \neq 0\}
\end{aligned}
\ \text{(see Equation 2.1).} \tag{2.7}
$$

Lemma 3 *The geometric dual $GD(C\ell)$ is feasible and there exists a feasible point $\delta > 0$.*

Proof. The proof is constructive. Set:

$$\delta_{ij} = \frac{\beta}{(S-N)N+N} \quad i \neq 0 \quad (i,j) \neq (1, S-N+1)$$

$$\delta_{0j} = \frac{1}{\beta}\left(\sum_{i \in p(j)} \delta_{ij}\right) \quad j = 1, \ldots, S \quad j \neq S-N+1$$

$$\delta_{0(S-N+1)} = 1 - \sum_{j \neq S-N+1}^{S} \delta_{0j}$$

$$\delta_{1(S-N+1)} = \frac{1}{\beta}\delta_{0(S-N+1)} \,.$$

This solution satisfies the hypothesis. $\hspace{4cm}$ Q.E.D. $\quad\square$

Lemma 4 *Program $Q(C\ell)$ has a solution.*

Proof. The proof follows directly from Lemmas 2, 3 and (Duffin et al. [5], page 81).

$\hspace{10cm}$ Q.E.D $\quad\square$

The Algorithm

Step (0): Choose $\lambda_i^{(0)} = 1$ $\hspace{4cm}$ $i = 1, \ldots, N$.

Step (k):

$$x_j^{(k)} = \left(\frac{\beta C_j}{\sum_{i=1}^{N} \lambda^{(k)} a_{ij}}\right)^{1/\beta+1} \quad j = 1, \ldots, S \tag{2.8}$$

$$\lambda_i^{(k+1)} = \begin{cases} \lambda_i^{(k)}[(1-\alpha^{(k)}) + \alpha^{(k)} g_i(x^{(k)})] \; i = 1, \ldots, N & (2.9a) \\[2mm] \text{we have fixed the value of } \alpha^{(k)} = 1 \text{ thus :} \\[2mm] \lambda_i^{(k)} g_i(x^{(k)}) \;\; i = 1, \ldots, N. & (2.9b) \end{cases}$$

Set $k \leftarrow (k+1)$ and go to step k.

3 Properties of the Algorithm Numerical Results

The λ's generated by the algorithm are an approximation of the Lagrange multipliers. Let $L(x, \lambda)$ be the Lagrangian function associated with $Q(C\ell)$

$$L(x, \lambda) = g_0(x) + \sum_{i=1}^{N}(g_i(x) - 1)\lambda_i \tag{3.1}$$

where $g_0(x)$ and $g_i(x)$ respectively denote the objective function and the linear constraints.

Lemma 5 *The saddle-point of the Lagrangian satisfies: (a) $x^* > 0$, (b) $\lambda^* > 0$.*

Proof. Since the program is convex, has a solution and satisfies Slater's condition (see Lemma 1) its Lagrangian possesses a saddle-point.

(a) Follows from Lemma 3.

(b) By solving $\partial L(x^*, \lambda)/\partial x_{(S-N+i)} = 0$ for λ^*, it follows that:

$$\lambda_i^* = C_{(S-N+i)}\beta(x_{S-N+i}^*)^{-(\beta+1)}/a_{i,(S-N+i)} > 0 \quad i = 1, \ldots, N$$

Q.E.D. $\square$

In the following lemmas we shall prove several properties of the algorithm.

Lemma 6 $x^{(k)}$, *Equation (2.8), is the solution of* $\min_{x>0} L(x, \lambda^{(k)})$, *Equation (3.1).*

Proof. This can be verified by solving the equation $\nabla_x L(x, \lambda^{(k)}) = 0$ for $x^{(k)}$.

Q.E.D. $\square$

By using the optimality conditions for geometric programming, Duffin et al. [5, page 80], it is possible to obtain additional insight into the method.

$$
\text{Set}: \quad
\left.
\begin{aligned}
\delta_{0,j}^{(k)} &= \frac{C_j(x_j^{(k)})^{-\beta}}{g_0(x^{(k)})} & j = 1, \ldots, S \\[2mm]
\delta_{i,j}^{(k)} &= \frac{\lambda_i^{(k)} a_{ij} x_j^{(k)}}{g_0(x^{(k)})} & i = 1, \ldots, N \\[2mm]
\gamma_i(\delta^{(k)}) &= \sum_{j\in q(i)} \delta_{ij}^{(k)} & j \in q(i) \,.
\end{aligned}
\right\} \tag{3.2}
$$

In this case $\delta_{ij}^{(k)}$ are dual feasible, Equations (2.4)–(2.6). The optimality conditions are:

1) dual feasibility, Equations (2.4), (2.5) and (2.6)

2) primal dual relations, Equation (3.2) and

3)

$$\gamma_i^* = \lambda_i^*/g_0(x^*) \quad i = 1, \ldots, N \,. \tag{3.3}$$

Of these three conditions, the first two are satisfied at each iteration. The third condition is not satisfied. Instead, one has:

$$\gamma_i(\delta^{(k)}) = \frac{\lambda_i^{(k)} g_i(x^{(k)})}{g_0(x^{(k)})} \quad i = 1, \ldots, N \; . \tag{3.4}$$

The updating formula is given by Equation (2.9):

$$\lambda_i^{(k+1)} = \lambda_i^{(k)} g_i(x^{(k)}) \quad i = 1, \ldots, N \; .$$

Thus, Equation (3.4) can be written as follows:

$$\gamma_i(\delta^{(k)}) = \lambda_i^{(k+1)} / g_0(x^{(k)}) \quad i = 1, \ldots, N \; . \tag{3.5}$$

If the algorithm converges, then the third optimality condition is satisfied (compare Equation (3.5) with (3.3)).

Lemma 7 *The Lagrange multipliers generated by the algorithm are bounded from above*

$$\lambda_i^{(k)} \le 1 \quad \forall k > 1 \; .$$

Proof. It follows from Equations (2.8) and (2.9) that

$$
\begin{aligned}
\lambda_i^{(k+1)} &= \lambda_i^{(k)} \sum_{j=1}^{S} a_{ij} \left[\frac{\beta C_j}{\sum_{u=1}^{N} a_{uj} \lambda_u^{(k)}} \right]^{1/(\beta+1)} \\
&= \sum_{j=1}^{S} (\beta C_j)^{1/(\beta+1)} \frac{a_{ij} \lambda_i^{(k)}}{\left[\sum_{u=1}^{N} a_{uj} \lambda_u^{(k)} \right]^{1/(\beta+1)}} \le \\
&\le \sum_{j=1}^{S} (\beta C_j)^{1/(\beta+1)} (a_{ij} \lambda_i^{(k)})^{\beta/\beta+1} \le (\lambda_i^{(k)})^{\beta/(\beta+1)}
\end{aligned}
\tag{3.6}
$$

(see Equation (2.2)).

Thus, if $\lambda_i^{(0)} = 1$ then $\lambda_i^{(k)} \le 1$ $i = 1, \ldots, N \forall k$. \hfill Q.E.D. $\square$

Lemma 8 $x^{(0)}$ *is a feasible solution of* $Q(C\ell)$.

Proof.

$$1 \ge \lambda_i^{(1)} = \lambda_i^{(0)} g_i(x^{(0)}) = g_i(x^{(k-1)}) \quad i = 1, \ldots, N \; \forall k.$$

Q.E.D.$\square$

Lemma 9 *The multipliers generated by the algorithm,* $\lambda^{(k)}$, *are bounded from below.*

Proof. It follows from Equations (2.1) and (2.8) that

$$(i) \quad x^{(k)}_{S-N+i} = \left[\frac{\beta C_{S-N+i}}{\lambda^{(k)}_i a_{i,(S-N+i)}} \right]^{1/(\beta+1)} \quad \forall k$$

$$(ii) \quad \lambda^{(k+1)}_i = \lambda^{(k)}_i g_i(x^{(k)}) \geq \lambda^{(k)}_i a_{i,(S-N+i)} x^{(k)}_{S-N+i} \; .$$

Thus,

$$x^{(k+1)}_{S-N+i} = \left[\frac{\beta C_{S-N+i}}{\lambda^{(k+1)}_i a_{i(S-N+i)}} \right]^{1/(\beta+1)} \leq \left[\frac{\beta C_{S-N+i}}{\lambda^{(k)}_i a^2_{i(S-N+i)} x^{(k)}_{S-N+i}} \right]^{1/(\beta+1)}$$

and

$$\frac{x^{(k+1)}_{S-N+i}}{x^{(k)}_{S-N+i}} \leq \left[\frac{1}{a_{i,(S-N+i)} x^{(k)}_{S-N+i}} \right]^{1/(\beta+1)} \; .$$

Therefore,

$$a_{i,(S-N+i)} x^{(k+1)}_{S-N+i} \leq \left[a_{i,(S-N+i)} x^{(k)}_{S-N+i} \right]^{\beta/\beta+1} \; .$$

However, from Lemma 8, it follows that

$$a_{i,(S-N+i)} x^{(0)}_{S-N+i} \leq g_i(x^{(0)}) \leq 1$$

and therefore,

$$(i) \quad a_{i,(S-N+i)} x^{(k)}_{S-N+i} \leq 1 \quad \forall k$$

$$(ii) \quad \lambda^{(k+1)}_i \geq \beta C_{S-N+i} a^{\beta}_{i,(S-N+i)} \quad \forall k \; .$$

<div align="right">Q.E.D.□</div>

The algorithm generates a sequence $\{\lambda^k\}_{k=0}^{\infty}$. It follows from Lemmas 7 and 9 that there exists a converging subsequence. However, it was not possible, in general, to prove convergence if α^k is fixed and set equal to 1. But, if the sequence converges it converges to the unique minimum of $Q(C\ell)$. It was possible to prove *local convergence*, i.e., there exists an $\epsilon > 0$ such that for $\lambda^0 : \|\lambda^* - \lambda^0\| < \epsilon$ the algorithm converges. The proof is based on Ostrowski's Theorem, Ortega and Rheinholdt [13, page 145] and is simple but tedious and therefore not included in the paper.

A somewhat different view of the method is obtained by evaluating the *Lagrangian* dual (in contrast to $(GD)(C\ell)$). This dual is concave and unconstrained, in the sense that a maximum exists in the strictly positive quadrant. It is given by:

$$D(C\ell) : \max_{\lambda>0} \mathcal{L}(\lambda) \stackrel{\triangle}{=} \max_{\lambda>0} L(x(\lambda), \lambda) = \max_{\lambda>0} \{ (\beta^{1/(\beta+1)} + \beta^{-\beta(\beta+1)}).$$

$$\cdot \left[\sum_{j=1}^{S} C^{1/(\beta+1)}_j \left(\sum_{i=1}^{N} a_{ij}\lambda_i \right)^{\beta/(\beta+1)} \right] - \sum_{i=1}^{N} \lambda_i \}. \tag{3.7}$$

This unconstrained dual was obtained after substituting $x(\lambda)$, Equation (2.8), into Equation (3.1), and performing several algebraic manipulations.

The algorithm may be viewed as an unconstrained *ascent* method in the dual space, (3.7), without line searches. For this, observe:

$$\partial \mathcal{L}(\lambda)/\partial \lambda_i = g_i(x(\lambda)) - 1 \quad i = 1, \ldots, N .$$

Given a point $\lambda^{(k)}$, the next point $\lambda^{(k+1)}$ is determined by:

$$\lambda^{(k+1)} = \lambda^{(k)} + \alpha^{(k)} d^{(k)} , \quad \text{(compare with Equation (2.9))} \quad (3.8)$$
$$\text{where: } (i) \quad d^{(k)} = \Lambda(k) \nabla^T \mathcal{L}(\lambda^{(k)}) , \ \lambda^{(k)} \quad (3.9)$$
$$\text{and } (ii) \quad \alpha^{(k)} = 1. \quad (3.10)$$

The matrix $\Lambda(k)$ is a diagonal matrix whose elements are $\lambda_i^{(k)}$, and $\alpha^{(k)}$ is the step size. The vector $d^{(k)}$ is an ascent direction, but does not coincide with the gradient:

$$\nabla \mathcal{L}(\lambda^{(k)}) , \ \lambda^{(k)}) d^{(k)} > 0 \forall k .$$

If the algorithm is modified, $\alpha^{(k)}$ is not kept constant (see Equations (2.9),(3.10)) and is given by:

$$\alpha^k \in \{\alpha^* : \mathcal{L}(\lambda^k + \alpha^* d^k) = max_{\alpha>0}\{\mathcal{L}(\lambda^k + \alpha d^k)\} \quad (3.12)$$

using Equation (2.9a) instead of Equation (2.9b). Then the algorithm generates a monotonic increasing sequence $\{\mathcal{L}(\lambda^k)\}_{k=0}^{\infty}$ that converges to the unique maximum of the dual, (3.7), and the algorithm converges. In this case the rate of convergence is at least Q linear, Ortega [14]; the proofs of the last two statements are omitted.

Several hundreds of problems have been solved. The coefficients ℓ_j were randomly chosen from the interval $(0, 100]$. The non-zero elements a_{ij} of the coefficient matrix were chosen similarly from the interval $(0, 50]$. The numerical results confirmed that the algorithm is simple and efficient and that the rate of convergence is at least linear. Since the algorithm always converged, we have used the *unmodified* algorithm thus eliminating the one-dimensional searches, Equation (3.12), therefore reducing the number of function evaluations. Thus, when viewed as a dual ascent method, it is probably superior to a Quasi-Newton approach.

The numerical results are shown in Figures 2 and 3, where

(i) N is the number of constraints.

(ii) S is the number of variables.

(iii) $\epsilon_k = \max_{1 \leq i \leq N} |1 - g_i(x^k)|$

The rate of convergence can be calculated from these experiments. From the figures, it follows that the number of iterations, k is a linear function of $\log \epsilon$, i.e.,

$$k = a + b \ \log_{10} \epsilon_k ,$$

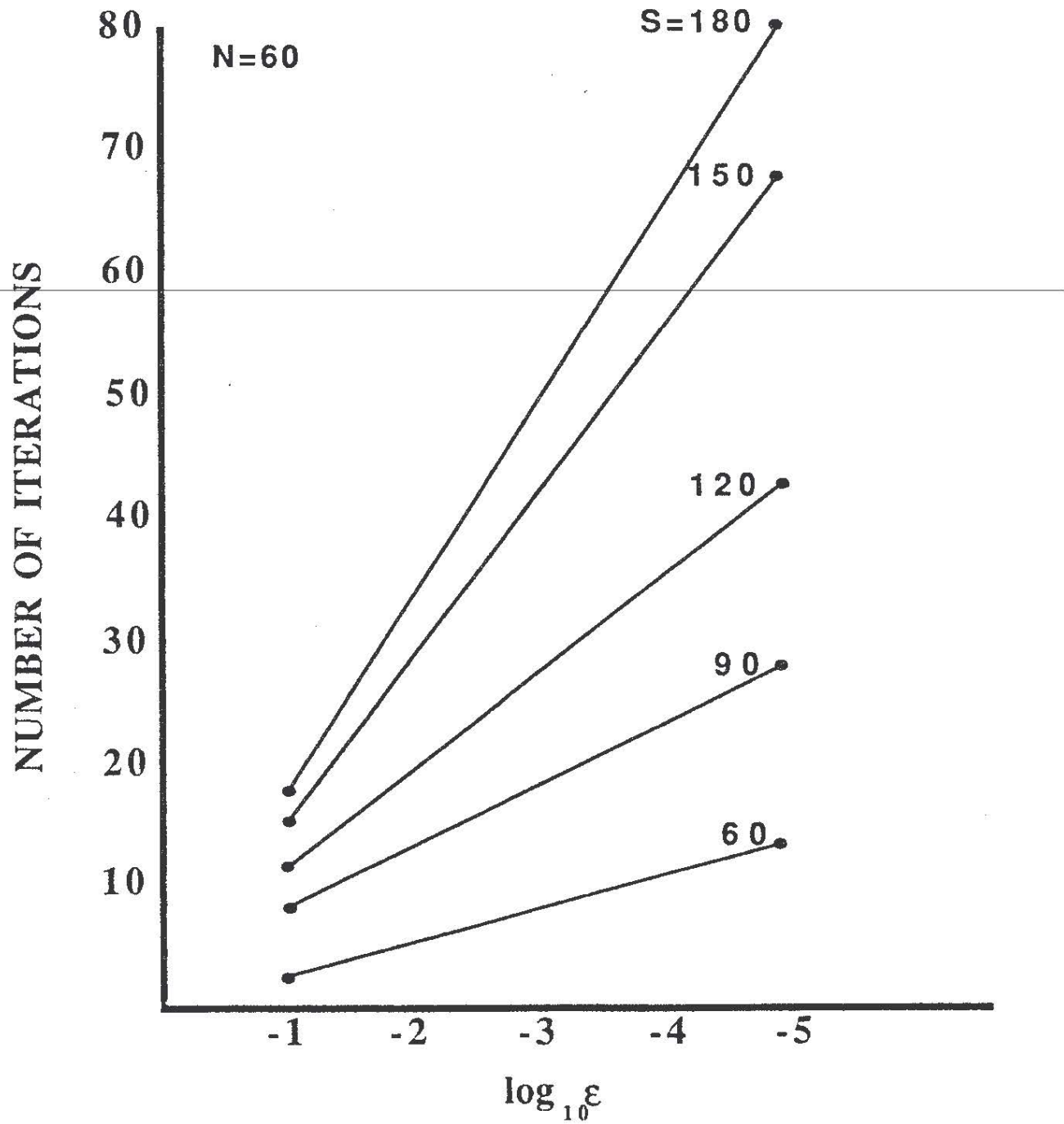where k is the iteration number, a and b are given constants. Observe that $b < 0$. Let:

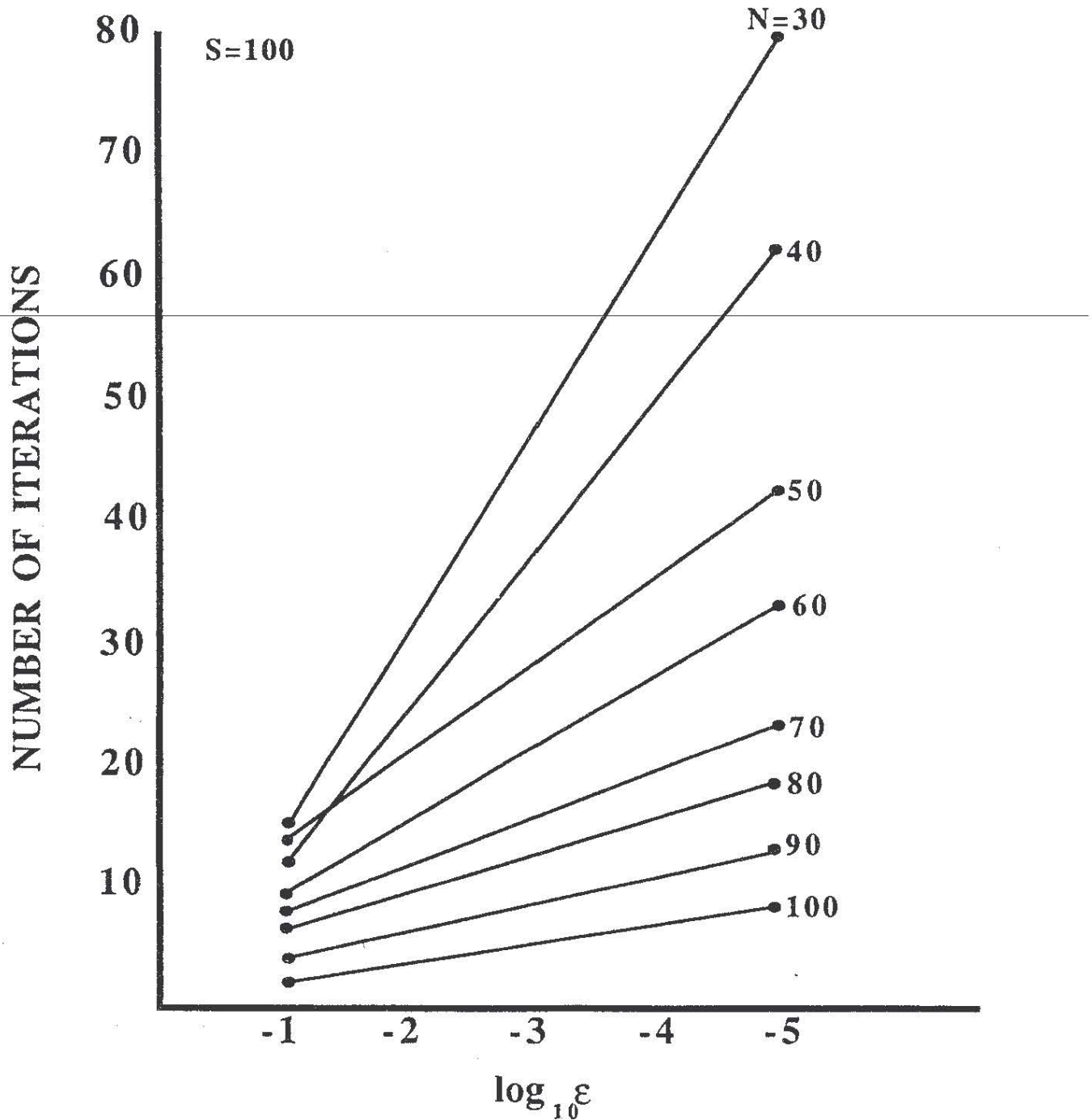<u>Fig. 2</u>: The dependence of the number of iterations on log ε and on S, for fixed number of constraints (N = 60).

Fig. 3: The dependence of the number of iterations on log ε and on N, for a fixed number of constraints (S = 100).

(i) $-a/b = \log_{10} \epsilon_0$

(ii) $1/b = \log_{10} E \quad E = 10^{1/b} < 1$

Thus,

$$\epsilon_k = E^k \, \epsilon_0$$

and

$$\epsilon_k = E\epsilon_{k-1} \quad \text{so that} \quad \epsilon_k/\epsilon_{k-1} = E < 1 \ .$$

Thus, it was found numerically that the rate of convergence is Q-linear and may be Q-superlinear, Ortega and Rheinholdt [13]. These experiments confirmed the theoretical results.

References

[1] E. ALPEROVITS and U. SHAMIR, Design of optimal water distribution system, Private Communication, 1977.

[2] American Petroleum Institute, Recommended practice for the design and installation of pressure-relieving systems in refineries, Part I, Design Report PI-RP 520, 1967.

[3] A. BEN-TAL and A. BARZILAI, A duality theory for a class of problems with essentially unconstrained duals, *Methods of Operations Research*, 45, 24–33 (1983).

[4] G. COHEN, Optimal control of water supply network, in S.G. Tzafetas (ed.), *Optimization and Control of Dynamic Operational Research Models*, North-Holland, 251–276.

[5] R.J. DUFFIN, E.L. PETERSON and C. ZENER, *Geometric Programming*, John Wiley and Sons, New York (1967).

[6] R.J. DUFFIN and E.L. PETERSON, Reversed geometric programs treated by harmonic means, *Indiana University Mathematical Journal*, 22, 531–550 (1972).

[7] R. EPP and A.G. FOWLER, Efficient code for steady-state flows in networks, *Journal of the Hydraulics Division, ASCE*, 96, 43–56, No. HY1, Proc. paper (January 1970).

[8] B. FUJIWARA, B. JENCHAIMAHAKOON, and N.C.P. EDIRISINGHE, A modified gradient method for optimal design of looped water distribution network, *Water Resources Research*, 23, 977–982 (1987).

[9] C.T. HANSEN, Optimization of large network for natural gas, Report NI-88-09, Danmarks Tekniske Højskole, 2800 Lyngby, Danmark (December 1988).

[10] E. KALLY, Computerized planning of the least cost water distribution network, *Water Sewage Works*, 121–127 (1972).

[11] B.A. MURTAGH, An approach to the optimal design of networks, *Chemical Engineering Science*, 27, 1131–1141 (1972).

[12] H.B. NIELSON, An efficient method for analyzing pipe networks, Report No. 362, DCAMM, The Technical University of Denmark, 2800 Lyngby, DK (1987).

[13] J.M. ORTEGA and Q. RHEINHOLDT, *Iterative Solutions of Nonlinear Equations in Several Variables*, Academic Press, New York (1970).

[14] J.M. ORTEGA, *Numerical Analysis: A Second Course*, Academic Press, New York (1972).

[15] URY PASSY, Signomial geometric programming: determining the global minimum", in Van Moeseke (ed.) *Mathematical Programs for Activity Analysis*, North Holland Publishing Company, 45–59 (1974).

[16] G.E. QUINDRY, E.D. BRILL and J.C. LIEBMAN, Comment on design of optimal water distribution system by E. Alperovits and U. Shamir, *Water Resources Research*, 15(6), 1651–1654 (December 1979).

[17] R. ROTHFARB, H. FRANK and D.M. ROSENBAUM, Optimal design of offshore natural gas pipeline systems, *Operations Research*, 18 (November–December, 1970).

[18] U. SHAMIR, Optimal design and operation of water distribution systems, *Water Resources Research*, 10(0), 27–36 (1974).

[19] T.M. WALSKI, State-of-the-art pipe network optimization, *ASCE, Spec. Conf. on Comp. Applications in Water Resources*, 559–568 (1985).

[20] D.J. WOOD and C.O.A. CHARLES, Hydraulic networks analysis using linear theory, *Journal of the Hydraulics Division, ASCE*, 98, No. HY7, Proc. paper, 1157–1170 (July 1972).