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COMBINATORIAL METHODS IN NETWORK RELIABILITY ANALYSIS [1)]

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ABSTRACT
In this paper, a stochastic network is an undirected graph with unreliable edges and absolutely reliable nodes. Its connectedness probability is determined by *reliability preserving network reduction*. The principle of this method consists in splitting the underlying deterministic graph of the stochastic network into two edge-disjoint subgraphs via a separating node set. One of the subgraphs is replaced with a simpler structured graph (replacement graph) in such a way that the interesting reliability criterion of the original stochastic network is retained. Special attention is given to the construction of suitable replacement graphs. The case of a 3-point separating node set is considered in more detail.

1. INTRODUCTION

Network reliability analysis arises in many important engineering areas, in particular communication networks, computer networks, monitoring and military systems as well as transportation and electrical power systems. Hence it is imperative that effective tools are being developed for the reliability analysis of complex networks with a general topological structure.

In this paper, a stochastic network is an undirected graph with unreliable edges and absolutely reliable nodes. The edges can be in two states: available (operating) or not. The terminology used throughout the paper refers to communication networks. Thus, if an edge is not available, then no direct transmission of information between its end nodes is possible. The paper only deals with the *connectedness probability* of a stochastic network, i.e. with the probability that there is a path between any node pair of the network, which only consists of

available edges. For the sake of convenience, the connectedness probability of a stochastic network is simply referred to as its *reliability*.

Basic Notation

$\tilde{\mathbf{G}}$	stochastic network under discussion
$\mathbf{G} = (\mathbf{N}, \mathbf{E})$	$\mathbf{G}$ underlying deterministic graph with node set $\mathbf{N} = \{1, 2, .., n\}$ and edge set $\mathbf{E}$
$\tilde{\mathbf{G}}'$	any stochastic network with underlying deterministic graph $\mathbf{G}'$
$R(\tilde{\mathbf{G}}'))$	connectedness probability (reliability) of $\tilde{\mathbf{G}}'$
$\mathbf{U}$, u	separating node set, cardinality of $\mathbf{U}$
$\varnothing$	empty set

2. NETWORK DECOMPOSITION

A subset $\mathbf{U}$ of $\mathbf{N}$ is said to be a *separating node set* of $\mathbf{G}$ if there exist two edge-disjoint subgraphs $\mathbf{G}^1 = (\mathbf{N}^1, \mathbf{E}^1)$ and $\mathbf{G}^2 = (\mathbf{N}^2, \mathbf{E}^2)$ of $\mathbf{G}$ such that

$$\mathbf{G} = \mathbf{G}^1 \cup \mathbf{G}^2, \quad \mathbf{G}^1 \cap \mathbf{G}^2 = (\mathbf{U}, \varnothing)$$

Figure 1 shows the splitting of a graph $\mathbf{G}$ with 11 nodes and 24 edges into two edge-disjoint subgraphs $\mathbf{G}^1$ and $\mathbf{G}^2$ by the separating node set $\mathbf{U} = \{1, 2, 3\}$ It is an intuitive approach to separately carry out reliability analysis for $\tilde{\mathbf{G}}^1$ and $\tilde{\mathbf{G}}^2$ and to combine the results to obtain $R(\tilde{\mathbf{G}})$. Pursuing this approach requires the introduction of some combinatorial concepts.

A *partition* π of $\mathbf{U}$ is a family of disjoint subsets of $\mathbf{U}$ the union of which is $\mathbf{U}$. The elements of π are called *blocks* Let $\mathbf{\Pi} = \{\pi_1, \pi_2, ..., \pi_B\}$ be the partition set of $\mathbf{U}$. $B = B(u)$ is the *Bell-number* of $\mathbf{U}$: $B(2) = 2$, $B(3) = 5$, $B(4) = 15$, $B(5) = 52$. $B(u)$ grows exponentially fast with increasing u. Partition π_j is a *refinement* of π_i if each block of π_j can be obtained by splitting a block of π_i. (Trivial splitting is allowed.)

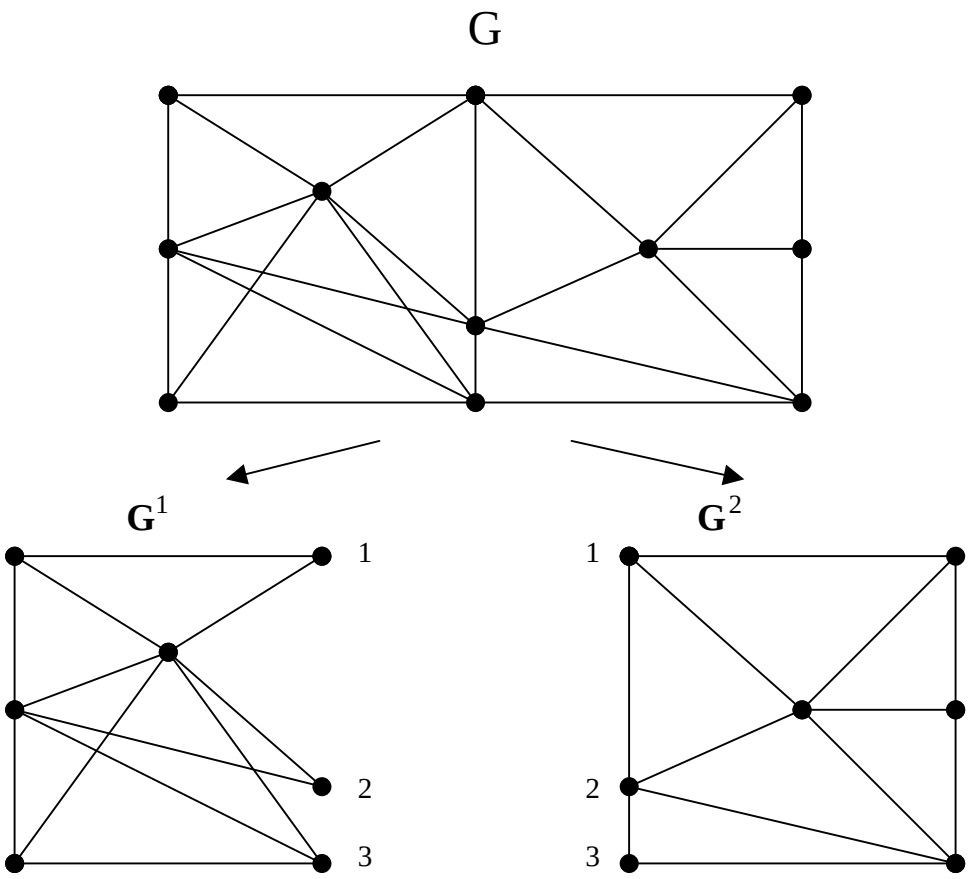

Figure 1 Decomposition of a graph with 11 nodes and 24 edges

In $\mathbf{\Pi}$ a partial order "≤" is introduced in the following way: The relation "$\pi_i \leq \pi_j$" holds if and only if π_j is a refinement of π_i. Thus, the *partition lattice* ($\mathbf{\Pi}$, ≤) is given. Let π_1 be its smallest and π_B its greatest element. Further, let $\pi_i \pi_j$ be the greatest partition satisfying both $\pi_i \pi_j \leq \pi_i$ and $\pi_i \pi_j \leq \pi_j$. Let $\mathbf{U}_j$ ($\mathbf{G}_j^i$) be that node set (graph) arising from $\mathbf{U}$ ($\mathbf{G}^i$) by fusing the nodes of each block of π_j into one node; $i = 1,2;\ j = 1,2,...,B.$ There is obviously a one-to-one correspondence between the blocks of π_j and the nodes of $\mathbf{U}_j$. Hence, in what follows, the elements of $\mathbf{U}_j$ will be identified with the corresponding blocks of π_j.The stochastic network $\tilde{\mathbf{G}}^1$ induces partitions of $\mathbf{U}$ in the following way: partition π_j is induced if the nodes belonging to a component of $\tilde{\mathbf{G}}^1$ belong to the same block of π_j. By assumption, the underlying deterministic graph $\mathbf{G}$ is connected.

Theorem The reliability of $\tilde{\mathbf{G}}$ is given by

$$R(\tilde{\mathbf{G}}) = \sum_{j=1}^{B} P_j(\tilde{\mathbf{G}}^1)\, R(\tilde{\mathbf{G}}_j^2), \tag{1}$$

where $P_j(\tilde{\mathbf{G}}^1)$ is the probability that $\tilde{\mathbf{G}}^1$ induces partition π_j and each component of $\tilde{\mathbf{G}}^1$ has a node in common with **U**.

Proof Let us introduce the following random events:

A each component of $\tilde{\mathbf{G}}^1$ has a node in common with **U**

A_j $\tilde{\mathbf{G}}^1$ induces partition π_j and each component of $\tilde{\mathbf{G}}^1$ has a node in common with **U**

Then $A_1, A_2, ..., A_B$ is a set of mutually exclusive random events with

$$A = A_1 \cup A_2 \cup \cdots \cup A_B$$

Therefore, $A_1, A_2, ..., A_B$ is a complete system of random events. Moreover, $R(\tilde{\mathbf{G}} \mid \overline{A}) = 0$, since, on condition $\overline{A}$, $\tilde{\mathbf{G}}^1$ contains at least one component, which is completely isolated from $\tilde{\mathbf{G}}^2$. On the other hand,

$$R(\tilde{\mathbf{G}} \mid \overline{A}_j) = R(\tilde{\mathbf{G}}_j^2),$$

since, given A_j, $\tilde{\mathbf{G}}$ is connected iff if $\tilde{\mathbf{G}}_j^2$ is connected, because in this case the components of $\tilde{\mathbf{G}}_j^1$ are connected by the components of $\tilde{\mathbf{G}}_j^2$ (and vice versa). Since $P(A_j) = P_j(\tilde{\mathbf{G}}_j^1)$, the proof of the theorem is finished by applying the total probability rule.

Of course, formula (1) is only useful if the probabilities $P_j(\tilde{\mathbf{G}}_j^1)$ are known. To establish a system of equations in the $P_j(\tilde{\mathbf{G}}_j^1)$, note that $\tilde{\mathbf{G}}_k^1$ is connected if and only if each component of $\tilde{\mathbf{G}}^1$ has a node in common with **U** and $P_j(\tilde{\mathbf{G}}_j^1)$ induces a partition π_j satisfying $\pi_j \pi_k = \pi_1$. Therefore, the probabilities $P_j(\tilde{\mathbf{G}}_j^1)$ satisfy

$$\sum_{j:\, \pi_j \pi_k = \pi_1} P_j(\tilde{\mathbf{G}}^1) = R(\tilde{\mathbf{G}}_k^1) \tag{2}$$

To simplify notation, let, for $i = 1, 2;\ k = 1, 2, ..., B$,

$$p_j^i = P_j(\tilde{\mathbf{G}}^i), \quad \mathbf{p}^i = (p_1^i, p_2^i, ..., p_B^i)^T$$

$$R_j^i = R(\tilde{\mathbf{G}}_j^i), \quad \mathbf{R}^i = (R_1^i, R_2^i, ..., R_B^i)^T$$

$$a_{jk} = \begin{cases} 1 & \text{if } \pi_j \pi_k = \pi_1 \\ 0 & \text{otherwise} \end{cases}$$

Note that $\mathbf{A} = ((a_{jk}))$ is the coefficient matrix of the system of linear equations (2). Hence, system (2) is equivalent to

$$\mathbf{A}\mathbf{p}^1 = \mathbf{R}^1 \tag{3}$$

Thus, formula (1) becomes

$$R(\tilde{\mathbf{G}}) = (\mathbf{p}^1)^T \mathbf{A}\mathbf{p}^2$$

(Of course, the roles of $\mathbf{G}^1$ and $\mathbf{G}^2$ can be exchanged.) From the theory of partition lattices it is well-known that the matrix **A** is regular (see, for instance, *Aigner* [1])). Hence, from (3), if $\mathbf{A}^{-1}$ denotes the inverse matrix of **A**,

$$\mathbf{p}^1 = \mathbf{A}^{-1}\mathbf{R}^1 \tag{4}$$

Thus, the decomposition formula obtains its final form:

$$R(\tilde{\mathbf{G}}) = (\mathbf{R}^1)^T \mathbf{A}^{-1}\mathbf{R}^2 \tag{5}$$

Note that the matrix **A** is a characteristic of the partition lattice ($\mathbf{\Pi}$, ≤) and depends only on the cardinality *u* of **U**. In particular, **A** does not depend on $\tilde{\mathbf{G}}^1$ and $\tilde{\mathbf{G}}^2$, respectively.

From the modeling point of view, determining $R(\tilde{\mathbf{G}})$ is equivalent to computing the availability of a binary coherent system. Hence, with respect to computational complexity, computing $R(\tilde{\mathbf{G}})$ is an NP-hard problem (*Ball* [2]). Thus, applying the decomposition formula (1) can be expected most efficient if **U** splits **G** into two subgraphs of about the same "size". The splitting approach has been applied in *Beichelt, Tittmann* [5] to the *K*-terminal reliability of a stochastic network. The proof of formula (1) given here is new.

Example 1 Let us consider the graph **G** which is split in Figure 1 by the separating node set **U** = {1, 2, 3} into two subgraphs $\mathbf{G}^1$ and $\mathbf{G}^2$. Let $\mathbf{\Pi} = \{\pi_1, \pi_2, ..., \pi_5\}$ be the partition set of **U** with

$$\pi_1 = \{123\}, \pi_2 = \{12,3\}, \pi_3 = \{13,2\}, \pi_4 = \{1,23\}, \pi_5 = \{1,2,3\} \tag{6}$$

Note that here and in what follows blocks are separated by commata. In particular, $\mathbf{U} = \pi_5$.

The corresponding matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ are

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

From (4),

$$p_1^1 = R_5^2$$

$$p_2^1 = \frac{1}{2}\left[-R_2^1 + R_3^1 + R_4^1 - R_5^1\right]$$

$$p_3^1 = \frac{1}{2}\left[+R_2^1 - R_3^1 + R_4^1 - R_5^1\right]$$

$$p_4^1 = \frac{1}{2}\left[+R_2^1 + R_3^1 - R_4^1 - R_5^1\right]$$

$$p_5^1 = \frac{1}{2}\left[2R_1^1 - R_2^1 - R_3^1 - R_4^1 + R_5^1\right]$$

i	π_i	R_i^1	R_i^2
1	{123}	0.983567	0.980261
2	{12,3}	0.938471	0.968360
3	{13,2}	0.943555	0.978309
4	{1,23}	0.972559	0.939692
5	{1,2,3}	0.892682	0.927445

Table 1 Numerical results for example 1 (Beichelt [3])

Table 1 shows the corresponding vectors $\mathbf{R}^1$ and $\mathbf{R}^2$ in case of the common edge availability p = 0.8. Formula (5) yields the reliability of $\tilde{\mathbf{G}}$:

$$R(\tilde{\mathbf{G}}) = 0.963096$$

Hence, $\tilde{\mathbf{G}}$ is not connected with probability $1 - R(\tilde{\mathbf{G}}) = 0.036904$.

3. NETWORK REDUCTION

Network reduction (network transformation) is a powerful tool for the reliability analysis of complex stochastic networks $\tilde{\mathbf{G}}$. Most of the known efficient (nonexponential) algorithms for computing network reliability criteria are based on network reduction. It is characterized by simplifying the topological structure of the underlying graph **G** by substituting a subgraph of **G** by one or more *replacement graphs* More exactly, *reliability preserving network reduction* involves three basic steps:

1) Decomposition of **G** by a separating node set **U**:

$$\mathbf{G} = \mathbf{G}^1 \cup \mathbf{G}^2, \quad \mathbf{G}^1 \cap \mathbf{G}^2 = (\mathbf{U}, \varnothing)$$

2) Generation of a graphs $\mathbf{H}^2$ by replacing $\mathbf{G}^2$ with a graphs $\mathbf{H}^{2,k}$, k = 1, 2,…r, satisfying

$$\mathbf{G}^1 \cap \mathbf{H}^{2,k} = (\mathbf{U}, \varnothing), \quad k = 1,2,...,r$$

3) The stochastic networks $\tilde{\mathbf{H}}^k$ belonging to $\mathbf{H}^k = \mathbf{G}^1 \cup \mathbf{H}^{2,k}$ are constructed in such a way that

$$R(\tilde{\mathbf{G}}) = T(R(\tilde{\mathbf{H}}^1), R(\tilde{\mathbf{H}}^2,..., R(\tilde{\mathbf{H}}^r)), \tag{7}$$

where $T(\cdot)$ is a one-to-one-function from $[0,1]^r$ on [0, 1].

As in $\tilde{\mathbf{G}}$, the *reduced* (*transformed*) *graphs* $\tilde{\mathbf{H}}^k$ are assumed to have absolutely reliable nodes. Moreover, the availabilities of the edges of $\tilde{\mathbf{G}}^1$ in $\tilde{\mathbf{H}}^k$ are the same as in $\tilde{\mathbf{G}}$; k = 1,2,…, r. Here only the case of a linear function

$$T(x_1, x_2,..., x_r) = h_1 x_1 + h_2 x_2 + \cdots + h_r x_r$$

is considered. Then the *reduction equation* (7) simplifies to

$$R(\tilde{\mathbf{G}}) = h_1 R(\tilde{\mathbf{H}}^1) + h_2 R(\tilde{\mathbf{H}}^2) + \cdots + h_r R(\tilde{\mathbf{H}}^r) \tag{8}$$

The factors h_k are called *reduction constants*. To obtain a system of equations in the h_k and in the unknown availabilities of the edges of $\tilde{\mathbf{H}}^{2,k}$ such that condition (8) is satisfied, formula (1) is applied to $\tilde{\mathbf{H}}^k = \tilde{\mathbf{G}}^1 \cup \tilde{\mathbf{H}}^{2,k}$:

$$R(\tilde{\mathbf{H}}^k) = \sum_{j=1}^{B} p_j^1 R(\tilde{\mathbf{H}}_j^{2,k}), \quad k = 1, 2,..., r \tag{9}$$

Combining (8) and (9) yields

$$R(\tilde{\mathbf{G}}) = \sum_{j=1}^{B} p_j^1 \sum_{k=1}^{r} h_k R(\tilde{\mathbf{H}}_j^{2,k}) \tag{10}$$

Comparing the coefficients of the p_j^1 in (1) and (10) yields

$$R_j^2 = \sum_{k=1}^{r} h_k R(\tilde{\mathbf{H}}_j^{2,k}); \quad j = 1,2,...,B \tag{11}$$

The reliabilities R_j^2 have to be determined by any suitable method, whereas the reliabilities $R(\tilde{\mathbf{H}}_j^{2,k})$ are given in terms of the unknown edge availabilities of $\tilde{\mathbf{H}}_j^2$.

If there is no redundant equation in (11), then two conditions are necessary for the existence of a unique solution of (11):

1) The total number of edges in the set of replacement graphs s satisfies $s \geq B - r$. If $s > B - r$; then the availabilities of $s - B + r$ edges have to be fixed. This should be done in such a way that solving (11) is facilitated.

2) Each partition of **U** can be generated by at least one of the stochastic replacement networks $\tilde{\mathbf{H}}^{2,1}, \tilde{\mathbf{H}}^{2,2}, ..., \tilde{\mathbf{H}}^{2,r}$.

In case $r = 1$, equations (8) and (11) simplify to

$$R(\tilde{\mathbf{G}}) = h\, R(\tilde{\mathbf{H}})$$

and

$$R_j^2 = h\, R(\tilde{\mathbf{H}}_j^2); \quad j = 1,2,...,B \tag{12}$$

Note: In what follows it is assumed that, for all $k = 1, 2, \ldots, r$, both in $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{H}}^k$ the indicator variables of the states of the edges are independent.

Example 2 Let graph **G** be split by a 3-point separating node set **U** = {1, 2, 3} into two edge-disjoint subgraphs $\mathbf{G}^1$ and $\mathbf{G}^2$. In this case it seems to be obvious to replace $\mathbf{G}^2$ with a single replacement graph ($r = 1$), namely a "star", i.e. a tree consisting of 3 edges which

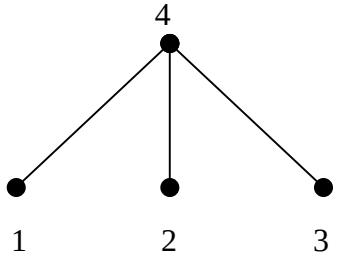

Figure 2 Replacement graph "star"

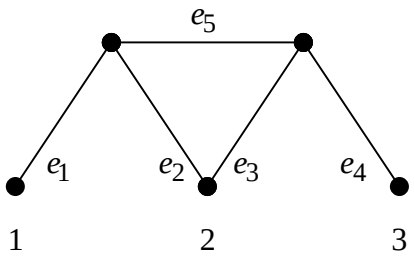

Figure 3 Replacement graph in example 2

have a node, say node 4, in common (Figure 2). This replacement graph generates all 5 partitions of **U**, but it has only 3 edges. Then (12) becomes a system of 5 equations in 4 unknowns. To generate another unknown parameter, the common node 4 can be assumed unreliable too. In this case, the reduced graph $\tilde{\mathbf{H}}$ would have both absolutely reliable and unreliable nodes, contradictory to our assumptions on $\tilde{\mathbf{H}}$. However, in case of the well-known *delta-star-reduction*, where a "delta" (triangle) is replaced with a star, the equations in (12) are linearly dependent in such a way that one of the equations is superfluous. Hence, in this special case a unique solution of (12) exists (see, for instance, *Beichelt* [4]. Unfortunately, in general, for a 3-point separating node set there is no tree with at least 4 edges which can serve as a replacement graph. (A replacement graph being a tree simplifies the structure of the system of equations (12) and, therefore, its solution.) Figure 3 shows a possible replacement graph with 5 edges $e_1, e_2, ..., e_5$ containing one cycle. Let p_i be the availability of edge e_i to be determined and $\bar{p}_i = 1 - p_i;\ i = 1,2,...,5.$ The availability of edge e_5 is fixed to be $p_5 = 1/2.$ Furthermore, let

$$a_j = 2R_j^2 / h;\quad j = 1,2,...,5$$

With **U** given by (4), the nonlinear system of equations (12) becomes (to verify, condition with respect to "e_5 available" and "e_5 not available"):

$$a_1 = 2 - \bar{p}_1\bar{p}_2 - \bar{p}_3\bar{p}_4$$

$$a_2 = (2 - \bar{p}_1\bar{p}_2 - \bar{p}_3)p_4$$

$$a_3 = (p_1 + p_4)(p_2 + p_3) - 2p_1p_2p_3p_4$$

$$a_4 = (2 - \bar{p}_2 - \bar{p}_3\bar{p}_4)p_1$$

$$a_5 = (p_2 + p_3)p_1p_4$$

There exists a unique solution $\{p_1, p_2, p_3, p_4\}$ of this system of equations. In view of its complicated structure, it cannot be given here. However, dependent on the numerical values of the R_j^2, the p_i need not satisfy conditions

$$0 \le p_i \le 1;\quad i = 1,2,...,5 \tag{13}$$

In case of the *star-delta-transformation* (a star is replaced with a triangle), this observation has been already made by *Rosenthal and Frisque* [6]. Nevertheless, also in what follows the p_i are referred to as "availabilities".

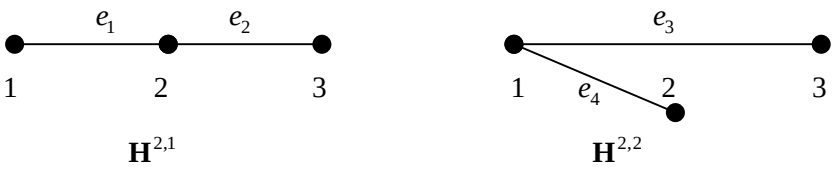

Figure 4 Replacement graphs in example 3

Example 3 As in the previous examples, let **U** = {1, 2, 3}. The subgraph $\mathbf{G}^2$ is replaced with the two graphs $\mathbf{H}^{2,1}$ and $\mathbf{H}^{2,2}$ depicted in Figure 4. Let p_1, p_2, p_3, p_4 be the availabilities of the edges e_1, e_2, e_3, e_4 and $p_4 = 0$. With $r_j = R_j^2;\ j = 1,2,...,5;$ the system of equations (11) becomes

$$\begin{aligned} r_1 &= h_1 + h_2 \\ r_2 &= h_1 p_2 + h_2 p_3 \\ r_3 &= h_1(p_1 + p_2 - p_1 p_2) \\ r_4 &= h_1 p_1 + h_2 p_3 \\ r_5 &= h_1 p_1 p_2 \end{aligned} \tag{14}$$

The solution is

$$p_1 = \frac{2r_5}{r_2 + r_3 + r_5 - r_4}$$

$$p_2 = \frac{2r_5}{r_3 + r_4 + r_5 - r_2}$$

$$p_3 = \frac{2r_5(r_3 + r_5 - r_2 - r_4)}{(r_2 + r_3 + r_5 - r_4)(r_3 + r_4 + r_5 - r_2) - 2r_1 r_5}$$

With p_1, p_2, p_3 known, h_1 is given by the last equation of (14) and h_2 by the first. Note that, for all replacement graphs belonging to a 3-point separating vertex set,

$$0 < r_5 \le r_2, r_3, r_4 \le r_1 < 1$$

Hence, p_1 and p_2 can be expected to satisfy condition (13), whereas p_3 is likely to fulfill (13) if $r_3 + r_5 > r_2 + r_4$.

Unfortunately, even for replacement graphs satisfying the necessary conditions 1) and 2) stated above, a solution of (11) need not exist. Hence, it is necessary to establish a list of

replacement graphs, which, dependent on the numerical values of the r_j, guarantee the existence of a solution of (11) with property (13). This task is still outstanding. Example 3 illustrates that the use of more than one replacement graph facilitates establishing such a list. The idea to use more than one replacement graph in one reduction step is due to *Tittmann* [7].

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