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An interior-point method for the Cartesian $P_*(\kappa)$-linear complementarity problem over symmetric cones

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Abstract

A novel primal-dual path-following interior-point algorithm for the Cartesian $P_*(\kappa)$-linear complementarity problem over symmetric cones is presented. The algorithm is based on a reformulation of the central path for finding the search directions. For a full Nesterov-Todd step feasible interior-point algorithm based on the new search directions, the complexity bound of the algorithm with small-update approach is the best-available bound.

Key words: Linear complementarity problem, full Nesterov-Todd step, small-update method, polynomial complexity.

1 Introduction

Let $(\mathcal{J}, \circ)$ be the Cartesian product of a finite number of Euclidean Jordan algebras, *i.e.*, $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2 \times \cdots \times \mathcal{J}_N$, with its cone of squares $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_N$, where $\mathcal{J}_j$ is an n_j-dimensional Euclidean Jordan algebra with $n = \sum_{j=1}^N n_j$ and $\mathcal{K}_j$ is the corresponding cone of squares to $\mathcal{J}_j$ with $\mathrm{rank}(\mathcal{J}_j) = r_j$ and $r = \sum_{j=1}^N r_j$. For a linear transformation $\mathcal{A} : \mathcal{J} \to \mathcal{J}$ and a $q \in \mathcal{J}$, the linear complementarity problem over symmetric cones (SCLCP) is to find $x, s \in \mathcal{J}$ such that

$$x \in \mathcal{K}, \quad s = \mathcal{A}(x) + q \in \mathcal{K}, \quad \text{and} \quad \langle x, s \rangle = 0,$$

where $\langle x, s \rangle$, denotes the Euclidean inner product.

The SCLCP includes a wide class of problems, namely, linear complementarity problem (LCP), second-order cone linear complementarity problem (SOCLCP) and semidefinite linear complementarity problem (SDLCP) as special cases. Moreover, the Karush Kuhn

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Tucker (KKT) condition of symmetric optimization (SCO) can be written in the form SCLCP [19]. For a comprehensive study on recent developments related to symmetric cone complementarity problems (SCCP), the reader is referred to [20]. SCLCP is called the Cartesian $P_*(\kappa)$-SCLCP if the linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property, *i.e.*,

$$(1+4\kappa)\sum_{\nu\in I_+(x)}\left\langle x^{(\nu)},[\mathcal{A}(x)]^{(\nu)}\right\rangle+\sum_{\nu\in I_-(x)}\left\langle x^{(\nu)},[\mathcal{A}(x)]^{(\nu)}\right\rangle\geq 0,$$

where κ is a nonnegative constant, $\nu\in\{1,2,\ldots,N\}$, and

$$I_+(x)=\{\nu:\langle x^{(\nu)},[\mathcal{A}(x)]^{(\nu)}\rangle>0\},\quad I_-(x)=\{\nu:\langle x^{(\nu)},[\mathcal{A}(x)]^{(\nu)}\rangle<0\}.$$

It is a straightforward extension of the $P_*(\kappa)$-matrix introduced by Kojima *et al.* [9]. If $\mathcal{K}=R^n_+$, which corresponds to $n_1=n_2=\ldots=n_N=1$ and $n=N$, the Cartesian $P_*(\kappa)$-SCLCP becomes the $P_*(\kappa)$-LCP. Moreover, it is evident that for $\kappa=0$, $P_*(0)$-SCLCP is the so-called monotone SCLCP [5]. The linear transformation $\mathcal{A}$ has the Cartesian P_*-property if it has the Cartesian $P_*(\kappa)$-property for some nonnegative κ, *i.e.*

$$P_*=\bigcup_{\kappa\geq 0}P_*(\kappa).$$

Interior-point methods (IPMs) for solving linear optimization (LO) were initiated by Karmarkar [8]. It is well known that the IPMs have polynomial complexity and are very efficient in practice. Among the variant IPMs, the primal-dual path-following methods are the most efficient from a computational point of view. These methods have used the so-called central path as a guideline to the optimal set. Kojima *et al.* [9] first proved the existence and uniqueness of the central path for the $P_*(\kappa)$-LCP and generalized the primal-dual interior-point algorithm for LO to the $P_*(\kappa)$-LCP. By using the techniques of Euclidean Jordan algebras, Faybusovich [5] first studied the interior-point algorithm for the monotone SCLCP and proved the existence and uniqueness of the central path. Gowda and Sznajder [6] presented some global uniqueness and solvability results for SCLCP. Luo and Xiu [10] first established a theoretical framework of path-following interior-point algorithms for the Cartesian $P_*(\kappa)$-SCLCP and proved the global convergence and the iteration complexities of the proposed algorithms. In addition to Faybusovich's results [3, 4], Rangarajan [11] proposed the first infeasible interior-point method (IIPM) for SCLCP. Yoshise [18] was the first to analyze IPMs for nonlinear complementarity problems over symmetric cones. Darvay [1] proposed a full-Newton step primal-dual path-following interior-point algorithm for LO. The search direction of his algorithm is introduced by using an algebraic equivalent transformation of the nonlinear equations which define the central path and then applying Newton's method for the new system of equations. Later on, Wang and Bai [15, 16] extended Darvay's algorithm for LO to SDO and symmetric optimization (SCO). Recently, Zhang and Xu [21] proposed a full-Newton step primal-dual interior-point algorithm for LO. The search directions of their algorithm is obtained by a new algebraic transformation of the centering equations and then applying Newton's method for the new system. In this paper, the Zhang and Xu's algorithm is generalised to the Cartesian $P_*(\kappa)$-SCLCP. A new method is proposed and analyzed for the Cartesian $P_*(\kappa)$-SCLCP. The algorithm uses the full Newton step in the methods of proximity measure for the first time. It is proved that this novel algorithm stops after at most $\mathcal{O}((1+2\kappa)\sqrt{r}\log\frac{r\mu^0}{\epsilon})$

iteration, where ϵ is the desired accuracy, μ^0 is the initial value of the barrier parameter, and r is the rank of the underlying Euclidean Jordan algebra. The complexity obtained here coincides with the best known bound, while tendering a simple analysis. It should be noted that the same algorithm can be introduced using the technique introduced in Darvay [1].

The remainder of this paper is organized as follows: In §2, after reviewing the properties of Euclidean Jordan algebra, the linear complementarity problem is generalized. In §3, the new search directions and new algorithm is presented. The complexity analysis for the algorithm based on the new search direction is given in §4.

2 Euclidean Jordan algebra

Some important results on Euclidean Jordan algebra and symmetric cones are presented in this section. For a more comprehensive study, the reader is referred to [2, 5, 14].

A Jordan algebra $\mathcal{J}$ is a finite dimensional vector space endowed with a bilinear map $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ satisfying the following properties for all $x, y \in \mathcal{J}$:

- $x \circ y = y \circ x$,
- $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$.

Moreover, a Jordan algebra $(\mathcal{J}, \circ)$ is called Euclidean if an associative inner product $\langle \cdot, \cdot \rangle$ is defined, *i.e.* for all $x, y, z \in \mathcal{J}$,

$$\langle x \circ y, z \rangle = \langle x, y \circ z \rangle.$$

A Jordan algebra has an identity element, if there exists a unique element $e \in \mathcal{J}$ such that $x \circ e = e \circ x = x$, for all $x \in \mathcal{J}$. The set $\mathcal{K} = \{x^2 : x \in \mathcal{J}\}$ is called the cone of squares of Euclidean Jordan algebra $(\mathcal{J}, \circ, \langle \cdot, \cdot \rangle)$. A cone is symmetric if and only if it is the cone of squares of some Euclidean Jordan algebra. An element $c \in \mathcal{J}$ is idempotent if $c \circ c = c$. Two elements x and y are orthogonal if $x \circ y = 0$. An idempotent c is primitive if it is nonzero and cannot be expressed by the sum of two other nonzero idempotents. A set of primitive idempotents $\{c_1, c_2, \ldots, c_k\}$ is called a Jordan frame if $c_i \circ c_j = 0$, for any $i \neq j \in \{1, 2, \ldots, k\}$ and $\sum_{i=1}^{k} c_i = e$. For any $x \in \mathcal{J}$, let r be the smallest positive integer such that $\{e, x, x^2, \ldots, x^r\}$ is linearly dependent; r is called the degree of x and is denoted by $\deg(x)$. The rank of $\mathcal{J}$, denoted by $\operatorname{rank}(\mathcal{J})$, is defined as the maximum of $\deg(x)$ over all $x \in \mathcal{J}$. The importance of a Jordan frame comes from the fact that any element of Euclidean Jordan algebra can be represented using some Jordan frame, as explained more precisely in the following spectral decomposition theorem.

Theorem 1 *(Theorem III.1.2 in [2]) Let $(\mathcal{J}, \circ, \langle \cdot, \cdot \rangle)$ be an Euclidean Jordan algebra with* $\operatorname{rank}(\mathcal{J}) = r$. *Then, for any $x \in \mathcal{J}$, there exists a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and real numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$ such that*

$$x = \sum_{i=1}^{r} \lambda_i(x) c_i.$$

The numbers $\lambda_i(x)$ (with their multiplicities) are the eigenvalues of x. Furthermore,

$$\operatorname{Tr}(x) = \sum_{i=1}^{r} \lambda_i(x) \quad \text{and} \quad \det(x) = \prod_{i=1}^{r} \lambda_i(x). \qquad \blacksquare$$

Since "$\circ$" is bilinear map, for every $x \in \mathcal{J}$, there exists a matrix $L(x)$ such that for every $y \in \mathcal{J}, x \circ y = L(x)y$. In particular, $L(x)e = x$, $L(x)^{-1}e = x^{-1}$ and $L(x)x = x^2$. For each $x \in \mathcal{J}$, define

$$P(x) := 2L(x)^2 - L(x^2),$$

where, $L(x)^2 = L(x)L(x)$. The map $P(x)$ is called the quadratic representation of $\mathcal{J}$, which is an essential concept in the theory of Jordan algebra and plays an important role in the analysis of interior-point algorithms.

The next lemma contains a result of crucial importance in the design of IPMs within the framework of Jordan algebras.

Lemma 1 *(Lemma 2.2 in [4]) Let $x, s \in \mathcal{K}$, then $\operatorname{Tr}(x \circ s) \geq 0$ and it follows that $\operatorname{Tr}(x \circ s) = 0$ if and only if $x \circ s = 0$.* $\blacksquare$

For any $x, y \in \mathcal{J}$, x and y are said to be operator commutable if $L(x)$ and $L(y)$ commute, *i.e.*, $L(x)L(y) = L(y)L(x)$. In other words, x and y are operator commutable if for all $z \in \mathcal{J}$, $x \circ (y \circ z) = y \circ (x \circ z)$ (see, for example, Schmieta & Alizadeh [12]).

Theorem 2 *(Lemma X.2.2 in [2]) The elements x and y, with $x, y \in \mathcal{J}$, are operator commutable if and only if they share a Jordan frame, that is*

$$x = \sum_{i=1}^{r} \lambda_i(x) c_i \quad \text{and} \quad y = \sum_{i=1}^{r} \lambda_i(y) c_i,$$

for Jordan frame $\{c_1, c_2, \ldots, c_r\}$. $\blacksquare$

For any $x, y \in \mathcal{J}$, define the canonical inner product of $x, y \in \mathcal{J}$ as

$$\langle x, y \rangle = \operatorname{Tr}(x \circ y),$$

and the Frobenius norm of x as

$$\|x\|_F = \sqrt{\langle x, x \rangle}.$$

From these definitions it follows that

$$\|x\|_F = \sqrt{\operatorname{Tr}(x^2)} = \sqrt{\sum_{i=1}^{r} \lambda_i^2(x)}.$$

Moreover, $\lambda_{\min}(x) \leq \|x\|_F$, $\lambda_{\max}(x) \leq \|x\|_F$ and $|\langle x, y \rangle| \leq \|x\|_F \|y\|_F$. The following lemma shows the existence and uniqueness of a scaling point w corresponding to any points $x, s \in \operatorname{int}\mathcal{K}$ such that $P(w)$ takes s into x, where $\operatorname{int}\mathcal{K}$ denotes the interior of $\mathcal{K}$.

Lemma 2 *(Lemma 3.2 in[3]) Let $x, s \in \mathrm{int}\mathcal{K}$. Then, there exists a unique $w \in \mathrm{int}\mathcal{K}$ such that $x = P(w)s$. Moreover,*

$$w = P(x^{\frac{1}{2}})\Big(P(x^{\frac{1}{2}})s\Big)^{-\frac{1}{2}}\Big[= P(s^{-\frac{1}{2}})\Big(P(s^{\frac{1}{2}})x\Big)^{\frac{1}{2}}\Big]. \qquad \blacksquare$$

The point w is called the Nesterov-Todd (NT)-scaling point of x and s. Two elements x and y, with $x, y \in \mathcal{J}$, are similar, as denoted by $x \sim y$, if and only if x and y share the same set of eigenvalues. Let $x \in \mathcal{K}$ if and only if $\lambda_i \geq 0$ and $x \in \mathrm{int}\mathcal{K}$ if and only if $\lambda_i > 0$, for all $i = 1, 2, \ldots, r$. Furthermore, x is positive semidefinite (positive definite) if $x \in \mathcal{K}$ ($x \in \mathrm{int}\mathcal{K}$). Since $\mathrm{Tr}(\cdot)$ is associative, *i.e.*, $\mathrm{Tr}(x \circ (y \circ z)) = \mathrm{Tr}((x \circ y) \circ z)$, it follows that

$$\langle L(x)y, z\rangle = \mathrm{Tr}((x \circ y) \circ z) = \mathrm{Tr}((y \circ x) \circ z) = \mathrm{Tr}(y \circ (x \circ z)) = \langle y, L(x)z\rangle,$$

showing that $L(x)$ is a self-adjoint operator. As the definition of $P(x)$ depends only on $L(x)$ and $L(x^2)$, both of which are self-adjoint, $P(x)$ is also self-adjoint.

Some results which are needed for the analysis of the algorithm are listed below.

Lemma 3 *(Proposition 21 in [12]) Let $x, s, u \in \mathrm{int}\mathcal{K}$, then*
(i) $P(x^{\frac{1}{2}})s \sim P(s^{\frac{1}{2}})x$.
(ii) $P\Big((P(u)x)^{\frac{1}{2}}\Big)P(u^{-1})s \sim P(x^{\frac{1}{2}})s$. $\blacksquare$

Lemma 4 *(Proposition 3.2.4 in [14]) Let $x, s \in \mathrm{int}\mathcal{K}$, and w be the scaling point of x and s, then*

$$\Big(P(x^{\frac{1}{2}})s\Big)^{\frac{1}{2}} \sim P(w^{\frac{1}{2}})s. \qquad \blacksquare$$

Lemma 5 *(Lemma 30 in [12]) Let $x, s \in \mathrm{int}\mathcal{K}$, then*

$$\|P(x)^{\frac{1}{2}}s - e\|_F \leq \|x \circ s - e\|_F. \qquad \blacksquare$$

Lemma 6 *(Theorem 4 in [13]) Let $x, s \in \mathrm{int}\mathcal{K}$, then*

$$\lambda_{\min}(P(x)^{\frac{1}{2}}s) \geq \lambda_{\min}(x \circ s). \qquad \blacksquare$$

Lemma 7 *(Lemmas 2.12 and 2.16 in [7]) Let $x, s \in \mathcal{J}$, then*
(i) $\|x^2\|_F \leq \|x\|_F^2$.
(ii) $\|x \circ s\|_F \leq \frac{1}{2}\|x^2 + s^2\|_F$. $\blacksquare$

Let $0 \leq \alpha \leq 1$, then define $x(\alpha) := x + \alpha\Delta x$ and $s(\alpha) := s + \alpha\Delta s$. The next lemma gives a condition for a feasible step-length $\bar{\alpha} > 0$ such that $x(\bar{\alpha}) \in \mathrm{int}\mathcal{K}$ and $s(\bar{\alpha}) \in \mathrm{int}\mathcal{K}$.

Lemma 8 *(Lemma 4.1 in [15]) Let $x, s \in \mathrm{int}\mathcal{K}$ and $x(\alpha) \circ s(\alpha) \in \mathrm{int}\mathcal{K}$ for $\alpha \in [0, \bar{\alpha}]$, then $x(\bar{\alpha}) \in \mathrm{int}\mathcal{K}$ and $s(\bar{\alpha}) \in \mathrm{int}\mathcal{K}$.* $\blacksquare$

2.1 The general case

In this section the definitions and properties stated so far are generalised to the case where the cone underlying the given $P_*(\kappa)$-SCLCP is the Cartesian product of N symmetric cones $\mathcal{K}_j$, where $N > 1$. Partition any vector $x \in \mathcal{J}$ according to the dimensions of the successive cones $\mathcal{K}_j$, such that

$$x = (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \in \mathcal{J} \Leftrightarrow x^{(j)} \in \mathcal{J}_j, \quad 1 \leq j \leq N \tag{1}$$

and

$$x = (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \in \mathcal{K} \Leftrightarrow x^{(j)} \in \mathcal{K}_j, \quad 1 \leq j \leq N. \tag{2}$$

The algebra $(\mathcal{J}, \diamond)$ is defined as a direct product of Jordan algebras $\mathcal{J}_j, j = 1, 2, \ldots, N$, with the product defined as follows

$$x \diamond s = (x^{(1)} \circ s^{(1)}, x^{(2)} \circ s^{(2)}, \ldots, x^{(N)} \circ s^{(N)}). \tag{3}$$

Obviously, if $e^{(j)} \in \mathcal{J}_j$ is the identity element in the Jordan algebra for the jth cone, then the vector

$$e = (e^{(1)}, e^{(2)}, \ldots, e^{(N)}), \tag{4}$$

is the identity element in $(\mathcal{J}, \diamond)$. It can easily be verified that $\mathrm{Tr}(e) = r$. The matrix $L(x)$ and the quadratic representation of $\mathcal{J}$ can be adjusted to

$$L(x) = \mathrm{diag}\big(L(x^{(1)}), L(x^{(2)}), \ldots, L(x^{(N)})\big), \tag{5}$$

and

$$P(x) = \mathrm{diag}\big(P(x^{(1)}), P(x^{(2)}), \ldots, P(x^{(N)})\big). \tag{6}$$

Let $x^{(j)} = \sum_{i=1}^{r_j} \lambda_i(x^{(j)}) c_i^{(j)}$ be the spectral decomposition of $x^{(j)} \in \mathcal{J}_j, 1 \leq j \leq N$. It follows from Theorem 1 that the spectral decomposition of $x \in \mathcal{J}$ can be adapted to

$$x = \Big(\sum_{i=1}^{r_1} \lambda_i(x^{(1)}) c_i^{(1)}, \sum_{i=1}^{r_2} \lambda_i(x^{(2)}) c_i^{(2)}, \ldots, \sum_{i=1}^{r_N} \lambda_i(x^{(N)}) c_i^{(N)}\Big). \tag{7}$$

Let $\{c_1^{(1)}, \ldots, c_{r_1}^{(1)}, \ldots, c_1^{(N)}, \ldots, c_{r_N}^{(N)}\}$ be the Jordan frame of $x \in \mathcal{J}$. The canonical inner product can be adjusted to

$$\langle x, s \rangle = \sum_{j=1}^{N} \langle x^{(j)}, s^{(j)} \rangle.$$

Furthermore,

$$\|x\|_F = \sqrt{\sum_{j=1}^{N} \|x^{(j)}\|_F^2} \quad \text{and} \quad \det(x) = \prod_{j=1}^{N} \det(x^{(j)}).$$

Let $x^{(j)}, s^{(j)} \in \operatorname{int}\mathcal{K}_j$ and $w^{(j)} \in \operatorname{int}\mathcal{K}_j$ be the NT-scaling point of $x^{(j)}$ and $s^{(j)}$, *i.e.*, $P(w^{(j)})s^{(j)} = x^{(j)}$, for each $j, 1 \leq j \leq N$. The scaling point of x and s in $\mathcal{K}$ is then defined by

$$w = (w^{(1)}, w^{(2)}, \ldots, w^{(N)}). \tag{8}$$

Since $P(w^{(j)})$ is symmetric and positive definite, for each $j, 1 \leq j \leq N$, the matrix

$$P(w) = \operatorname{diag}\big(P(w^{(1)}), P(w^{(2)}), \ldots, P(w^{(N)})\big), \tag{9}$$

is symmetric and positive definite as well and represents an automorphism of $\mathcal{K}$ such that

$$P(w)s = \left(P(w^{(1)})s^{(1)}, P(w^{(2)})s^{(2)}, \ldots, P(w^{(N)})s^{(N)}\right) = (x^{(1)}, x^{(2)}, \ldots, x^{(N)}) = x.$$

Therefore, $P(w)$ can be used to re-scale x and s to the same vector

$$v = (v^{(1)}, v^{(2)}, \ldots, v^{(N)}). \tag{10}$$

Finally, define

$$\lambda_{\min}(v) = \min\{\lambda_{\min}(v^{(j)}) : 1 \leq j \leq N\}$$

and

$$\lambda_{\max}(v) = \max\{\lambda_{\max}(v^{(j)}) : 1 \leq j \leq N\}.$$

The concept of the central path can also be extended to the Cartesian $P_*(\kappa)$-SCLCP. The existence and uniqueness of the central path for the Cartesian $P_*(\kappa)$-SCLCP were discussed by Luo and Xiu [10]. In developing the results it is assumed that the Cartesian $P_*(\kappa)$-SCLCP satisfies the interior-point condition (IPC), *i.e.*, there exists $x^0, s^0 \in \operatorname{int}\mathcal{K}$ such that $s^0 = \mathcal{A}(x^0) + q$ [10].

The basic idea of interior-point methods (IPMs) is to replace the complementarity condition $x \diamond s = 0$, for the Cartesian $P_*(\kappa)$-SCLCP, by the parameterized equation $x \diamond s = \mu e$, with $\mu > 0$. Thus, one may consider

$$s = \mathcal{A}(x) + q, \qquad x, s \in \operatorname{int}\mathcal{K}, \tag{11}$$

$$x \diamond s = \mu e. \tag{12}$$

For each $\mu > 0$, the system (11)–(12) has a unique solution $(x(\mu), s(\mu))$ (under given assumptions), and $(x(\mu), s(\mu))$ is called the μ-center of the Cartesian $P_*(\kappa)$-SCLCP. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of the Cartesian $P_*(\kappa)$-SCLCP. If $\mu \to 0$, then the limit of the central path exists, and since the limit points satisfy the complementarity condition $x \diamond s = 0$, the limit yields an optimal solution of the Cartesian $P_*(\kappa)$-SCLCP.

3 A new search directions and algorithm

The natural way to define a search direction is to follow the Newton approach and to linearize equation (12). This leads to the following system:

$$\begin{aligned} \mathcal{A}(\Delta x) - \Delta s &= 0 && (13)\\ x \diamond \Delta s + s \diamond \Delta x &= \mu e - x \diamond s. && (14) \end{aligned}$$

Due to the fact that $L(x)L(s) \neq L(s)L(x)$ in general, the system (13)–(14) does not always have a unique solution. It is known that this difficulty can be resolved by applying a scaling scheme. This is given in the following lemma.

Lemma 9 *(Lemma 28 in [12]) Let $u \in \mathrm{int}\mathcal{K}$. Then*

$$x \circ s = \mu e \Leftrightarrow P(u)x \circ P(u)^{-1}s = \mu e. \qquad \blacksquare$$

Replacing equation (12) by $P(u)x \diamond P(u^{-1})s = \mu e$, and applying the Newton method, it is obtained that

$$\begin{aligned} \mathcal{A}(\Delta x) - \Delta s &= 0, && (15)\\ P(u)\Delta x \diamond P(u)^{-1}s + P(u)x \diamond P(u)^{-1}\Delta s &= \mu e - P(u)x \diamond P(u)^{-1}s. && (16) \end{aligned}$$

The scaling point $u = w^{-\frac{1}{2}}$ is the focus, where w is the NT-scaling point of x and s as defined in Lemma 2. Define v as

$$v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} \left[= \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}} \right]. \tag{17}$$

Note that $x \diamond s = \mu e$ if and only if $v = e$ (Proposition 5.7.2 in [14]). Therefore, it is obtained that

$$v = e \Leftrightarrow \mu v = \mu e.$$

Combining the relation above with the system (15)–(16), it follows that

$$\begin{aligned} \mathcal{A}(\Delta x) - \Delta s &= 0, && (18)\\ P(u)\Delta x \diamond P(u)^{-1}s + P(u)x \diamond P(u)^{-1}\Delta s &= \mu v - P(u)x \diamond P(u)^{-1}s. && (19) \end{aligned}$$

Denote

$$\overline{\mathcal{A}} := P(w)^{\frac{1}{2}}\mathcal{A}P(w)^{\frac{1}{2}}, \quad d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}}, \quad \text{and} \quad d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}. \tag{20}$$

Note that the linear transformation $\overline{\mathcal{A}}$ has the Cartesian $P_*(\kappa)$-property if the linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property (Proposition 3.4 in [10]). Using (20), the system (18)–(19) turns to

$$\begin{aligned} \overline{\mathcal{A}}(d_x) - d_s &= 0, && (21)\\ d_x + d_s &= e - v. && (22) \end{aligned}$$

An important ingredient of this paper is that the analysis is different from the previous works, which is key to proving the polynomial complexity of the new algorithm. For this purpose, the search directions d_x and d_s are obtained by solving system (21)–(22), so Δx and Δs can be computed via (20). The new iterate is obtained by taking a full NT-step as

$$x_+ = x + \Delta x, \qquad s_+ = s + \Delta s. \tag{23}$$

For more analysis of the algorithm, define a norm-based proximity measure $\sigma(x, s; \mu)$ as follows:

$$\sigma(x, s; \mu) := \sigma(v) := \|e - v\|_F. \tag{24}$$

It is assumed that a pair (x^0, s^0) with $x^0 \in \text{int}\mathcal{K}$ and $s^0 \in \text{int}\mathcal{K}$ is given that is close to $(x(\mu), s(\mu))$ for some $\mu = \mu^0$ in the sense of the proximity measure $\sigma(x^0, s^0; \mu^0)$. The steps of the algorithm are summarized as Algorithm 1.

Algorithm 1: A novel algorithm for the Cartesian $P_*(\kappa)$ — SCLCP

Input : Accuracy parameter $\epsilon > 0$, barrier update parameter $0 < \theta < 1$, threshold parameter $0 < \tau < 1$ and a feasible solution x^0, s^0 and $\mu^0 > 0$ such that $\sigma(x^0, s^0; \mu^0) \leq \tau$

$x \leftarrow x^0$, $s \leftarrow s^0$, $\mu \leftarrow \mu^0$;
while $\langle x, s \rangle \geq \epsilon$ **do**
  solve system (21)–(22) and via (20) to obtain $(\Delta x, \Delta s)$;
  $x_+ \leftarrow x + \Delta x$;
  $s_+ \leftarrow s + \Delta s$;
  $\mu \leftarrow (1 - \theta)\mu$;

4 Analysis

In this section, the effects of a full NT-step and a μ-update for the quantity of the proximity measure are described. A proof that the algorithm is well defined is also supplied. It is also proven that the Cartesian $P_*(\kappa)$-SCLCP problem can be solved by this algorithm in polynomial-time.

4.1 Some basic results

Before dealing with the analysis of the algorithm, two needed lemmas are required. Since the linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property, it is obtained that

$$(1 + 4\kappa) \sum_{j \in J_+(\Delta x)} \left\langle \Delta x^{(j)}, [\mathcal{A}(\Delta x)]^{(j)} \right\rangle + \sum_{j \in J_-(\Delta x)} \left\langle \Delta x^{(j)}, [\mathcal{A}(\Delta x)]^{(j)} \right\rangle \geq 0, \tag{25}$$

where

$$J_+(\Delta x) = \{j : \langle \Delta x^{(j)}, [\mathcal{A}(\Delta x)]^{(j)} \rangle > 0\}, \quad J_-(\Delta x) = \{j : \langle \Delta x^{(j)}, [\mathcal{A}(\Delta x)]^{(j)} \rangle < 0\}.$$

It follows from (20) that

$$\langle d_x, d_s\rangle = \frac{\langle \Delta x, \Delta s\rangle}{\mu}.$$

Thus, equation (25) can be rewritten as

$$(1+4\kappa)\sum_{j\in J_+}\langle d_x^{(j)}, d_s^{(j)}\rangle + \sum_{j\in J_-}\langle d_x^{(j)}, d_s^{(j)}\rangle \geq 0, \tag{26}$$

where $J_+ = \{j : \langle \Delta x^{(j)}, \Delta s^{(j)}\rangle > 0,\ 1 \leq j \leq N\}$ and $J_- = \{j : \langle \Delta x^{(j)}, \Delta s^{(j)}\rangle < 0,\ 1 \leq j \leq N\}$.

Lemma 10 *The Frobenius norm of product of the search directions will always be less than or equal to $\frac{1+2\kappa}{2}$ times of square of the proximity measure $\sigma(v)$, i.e.,*

$$\|d_x \diamond d_s\|_F \leq \frac{1+2\kappa}{2}\sigma(v)^2.$$

Proof: It is known that

$$\frac{1}{4}\sum_{j\in J_+}\|d_x^{(j)} + d_s^{(j)}\|_F^2 - \sum_{j\in J_+}\langle d_x^{(j)}, d_s^{(j)}\rangle = \frac{1}{4}\sum_{j\in J_+}\|d_x^{(j)} - d_s^{(j)}\|_F^2 \geq 0.$$

This implies, by equations (21)–(22) and (24), that

$$\sum_{j\in J_+}\langle d_x^{(j)}, d_s^{(j)}\rangle \leq \frac{1}{4}\sum_{j\in J_+}\|d_x^{(j)} + d_s^{(j)}\|_F^2 \leq \frac{1}{4}\sum_{j\in J}\|d_x^{(j)} + d_s^{(j)}\|_F^2 = \frac{1}{4}\|d_x + d_s\|_F^2 = \frac{1}{4}\sigma(v)^2.$$

It follows immediately from (26) and the above relation that

$$-\sum_{j\in J_-}\langle d_x^{(j)}, d_s^{(j)}\rangle \leq (1+4\kappa)\sum_{j\in J_+}\langle d_x^{(j)}, d_s^{(j)}\rangle \leq \frac{1+4\kappa}{4}\sigma(v)^2.$$

On the other hand

$$\begin{aligned}\sigma(v)^2 = \|d_x + d_s\|_F^2 &= \|d_x\|_F^2 + \|d_s\|_F^2 + 2\Big(\sum_{j\in J_+}\langle d_x^{(j)}, d_s^{(j)}\rangle + \sum_{j\in J_-}\langle d_x^{(j)}, d_s^{(j)}\rangle\Big)\\ &\geq \|d_x\|_F^2 + \|d_s\|_F^2 - 8\kappa\sum_{j\in J_+}\langle d_x^{(j)}, d_s^{(j)}\rangle\\ &\geq \|d_x\|_F^2 + \|d_s\|_F^2 - 2\kappa\sigma(v)^2.\end{aligned}$$

From this, it can be deduced that

$$\|d_x\|_F^2 + \|d_s\|_F^2 \leq (1+2\kappa)\sigma(v)^2.$$

Therefore, by using $\sum_{i=1}^n a_i^2 \leq \left(\sum_{i=1}^n a_i\right)^2$ for all $a_i \geq 0$, Lemma 7 and the triangle inequality it is obtained that

$$\begin{aligned}
\|d_x \diamond d_s\|_F^2 = \sum_{j=1}^N \|d_x^{(j)} \circ d_s^{(j)}\|_F^2 &\leq \Big(\sum_{j=1}^N \|d_x^{(j)} \circ d_s^{(j)}\|_F\Big)^2 \\
&\leq \Big(\sum_{j=1}^N \frac{\|d_x^{(j)}\|_F^2 + \|d_s^{(j)}\|_F^2}{2}\Big)^2 \\
&= \Big(\frac{\|d_x\|_F^2 + \|d_s\|_F^2}{2}\Big)^2 \\
&\leq \Big(\frac{(1+2\kappa)\sigma(v)^2}{2}\Big)^2.
\end{aligned}$$

This completes the proof. ■

The next lemma gives a fundamental property about the proximity measure $\sigma(v)$.

Lemma 11 *The eigenvalues of vectors $v^{(j)}$ for $j = 1, 2, \ldots, N$ will always be between $1-\sigma(v)$ and $1+\sigma(v)$, i.e.,*

$$1 - \sigma(v) \leq \lambda_i(v^{(j)}) \leq 1 + \sigma(v), \;\; i = 1, 2, \ldots, r_j, \;\; j = 1, 2, \ldots, N.$$

Proof: By the definition of $\sigma(v)$ (cf. (24)), it follows that

$$\begin{aligned}
\sigma(v)^2 = \|e - v\|_F^2 &= \sum_{j=1}^N \|e^{(j)} - v^{(j)}\|_F^2 \\
&= \sum_{j=1}^N \operatorname{Tr}\Big((e^{(j)} - v^{(j)})^2\Big) \\
&= \sum_{j=1}^N \sum_{i=1}^{r_j} \Big(1 - \lambda_i(v^{(j)})\Big)^2.
\end{aligned}$$

The above expression implies that

$$|1 - \lambda_i(v^{(j)})| \leq \sigma(v), \quad i = 1, 2, \ldots, r_j, \quad j = 1, 2, \ldots, N,$$

which completes the proof. ■

4.2 Properties of the full-NT step

Using equations (20) and (23), for each $1 \leq j \leq N$, it is obtained that

$$\begin{aligned}
x_+^{(j)} &= x^{(j)} + \Delta x^{(j)} = \sqrt{\mu} P(w^{(j)})^{\frac{1}{2}} (v^{(j)} + d_x^{(j)}), \\
s_+^{(j)} &= s^{(j)} + \Delta s^{(j)} = \sqrt{\mu} P(w^{(j)})^{-\frac{1}{2}} (v^{(j)} + d_s^{(j)}).
\end{aligned} \tag{27}$$

By using the second equation of (24), it follows that

$$(v^{(j)}+d_x^{(j)})\circ(v^{(j)}+d_s^{(j)}) = (v^{(j)})^2+v^{(j)}\circ(d_x^{(j)}+d_s^{(j)})+d_x^{(j)}\circ d_s^{(j)} \\ = v^{(j)}+d_x^{(j)}\circ d_s^{(j)}, \quad j=1,2,\ldots,N. \tag{28}$$

Since $P(w)^{\frac{1}{2}}$ and $P(w)^{-\frac{1}{2}}$ are automorphisms of $\mathrm{int}\mathcal{K}$ (Theorem III.2.1 in [2]), x_+ and s_+ belong to $\mathrm{int}\mathcal{K}$ if and only if $v+d_x$ and $v+d_s$ belong to $\mathrm{int}\mathcal{K}$. The main aim of this subsection is to find conditions for strict feasibility of the full-NT step.

Lemma 12 *The full-NT step is strictly feasible if* $v+d_x\diamond d_s\in\mathrm{int}\mathcal{K}$.

Proof: Introduce a step length α with $\alpha\in[0,1]$ and for each $j=1,2,\ldots,N$, define

$$(v^{(j)})_x^{\alpha}=v^{(j)}+\alpha d_x^{(j)}, \qquad (v^{(j)})_s^{\alpha}=v^{(j)}+\alpha d_s^{(j)}.$$

It thus follows that $(v^{(j)})_x^0=v^{(j)}$, $(v^{(j)})_s^0=v^{(j)}$, $(v^{(j)})_x^1=v^{(j)}+d_x^{(j)}$ and $(v^{(j)})_s^1=v^{(j)}+d_s^{(j)}$. From equation (28), it follows that

$$(v^{(j)})_x^{\alpha}\circ(v^{(j)})_s^{\alpha} = (v^{(j)}+\alpha d_x^{(j)})\circ(v^{(j)}+\alpha d_s^{(j)}) \\ = (v^{(j)})^2+\alpha v^{(j)}\circ(d_x^{(j)}+d_s^{(j)})+\alpha^2 d_x^{(j)}\circ d_s^{(j)} \\ = (1-\alpha)(v^{(j)})^2+\alpha v^{(j)}+\alpha^2 d_x^{(j)}\circ d_s^{(j)}. \tag{29}$$

If $v+d_x\diamond d_s\in\mathrm{int}\mathcal{K}$, then, for $j=1,2,\ldots,N$, $v^{(j)}+d_x^{(j)}\circ d_s^{(j)}\in\mathrm{int}\mathcal{K}_j$ (cf. (2)) and it follows that $d_x^{(j)}\circ d_s^{(j)}\succ_{\mathcal{K}_j}-v^{(j)}$. Substituting this into equation (29), the result is

$$(v^{(j)})_x^{\alpha}\circ(v^{(j)})_s^{\alpha}\succ_{\mathcal{K}_j}(1-\alpha)(v^{(j)})^2+\alpha v^{(j)}-\alpha^2 v^{(j)}=(1-\alpha)v^{(j)}(\alpha+v^{(j)}).$$

If $0\le\alpha\le1$, then $(v^{(j)})_x^{\alpha}\circ(v^{(j)})_s^{\alpha}\succ_{\mathcal{K}_j}0$ $\left((v^{(j)})_x^{\alpha}\circ(v^{(j)})_s^{\alpha}\in\mathrm{int}\mathcal{K}_j\right)$, for $j=1,2,\ldots,N$. From Lemma 8, it follows that $(v^{(j)})_x^1=v^{(j)}+d_x^{(j)}\in\mathrm{int}\mathcal{K}j$ and $(v^{(j)})_s^1=v^{(j)}+d_s^{(j)}\in\mathrm{int}\mathcal{K}_j$, for each $j=1,2,\ldots,N$. Hence, the result of the lemma holds. ■

Corollary 1 *The new iterates* (x_+,s_+) *are strictly feasible if*

$$\|d_x\diamond d_s\|_F<\lambda_{\min}(v).$$

Proof: For $i=1,2,\ldots,r_j, j=1,2,\ldots,N$, it follows that

$$\lambda_i(v^{(j)}+d_x^{(j)}\circ d_s^{(j)})\ge\lambda_{\min}(v^{(j)}+d_x^{(j)}\circ d_s^{(j)})\ge\lambda_{\min}(v^{(j)})-\|d_x^{(j)}\circ d_s^{(j)}\|_F,$$

where the second inequality is followed by Lemma 14 in [12]. By Lemma 12, x_+ and s_+ are strictly feasible if $v+d_x\diamond d_s\in\mathrm{int}\mathcal{K}$, *i.e.*, for each $j=1,2,\ldots,N, v^{(j)}+d_x^{(j)}\circ d_s^{(j)}\in\mathrm{int}\mathcal{K}_j$. This holds if $\lambda_{\min}(v^{(j)})>\|d_x^{(j)}\circ d_s^{(j)}\|_F$. This last relation certainly holds if $\lambda_{\min}(v)>\|d_x\diamond d_s\|_F$, because for each $j=1,2,\ldots,N$, it follows that

$$\lambda_{\min}(v^{(j)})\ge\lambda_{\min}(v)>\|d_x\diamond d_s\|_F\ge\|d_x^{(j)}\circ d_s^{(j)}\|_F.$$

Therefore, the proof is complete. ■

Lemma 13 *Let $\sigma(v)$ be defined as (24), and $(x,s)\in \mathrm{int}\mathcal{K}\times\mathrm{int}\mathcal{K}$. If $\sigma(v)<\frac{2}{1+\sqrt{3+4\kappa}}$, then the full-NT step for the Cartesian $P_*(\kappa)$-SCLCP is strictly feasible, i.e., $(x_+,s_+)\in \mathrm{int}\mathcal{K}\times\mathrm{int}\mathcal{K}$.*

Proof: By Lemma 10, it follows that

$$\|d_x\diamond d_s\|_F\le\frac{1+2\kappa}{2}\sigma(v)^2. \tag{30}$$

By Corollary 1, the full-NT step is strictly feasible if $\|d_x\diamond d_s\|_F<\lambda_{\min}(v)$. This last inequality certainly holds, by Lemma 11 and (30), if

$$\frac{1+2\kappa}{2}\sigma(v)^2<1-\sigma(v),$$

which leads to $\sigma(v)<\frac{2}{1+\sqrt{3+4\kappa}}$. This completes the proof. ∎

The next lemma gives the effect of full-NT step duality gap.

Lemma 14 *If $\sigma(v)<\frac{2}{1+\sqrt{3+4\kappa}}$, then*

$$\langle x_+,s_+\rangle<2\mu r\left(\frac{3+\sqrt{3+4\kappa}}{1+\sqrt{3+4\kappa}}\right).$$

Proof: Due to (27), it can be written that

$$\langle x_+^{(j)},s_+^{(j)}\rangle=\langle\sqrt{\mu}P(w^{(j)})^{\frac12}(v^{(j)}+d_x^{(j)}),\sqrt{\mu}P(w^{(j)})^{-\frac12}(v^{(j)}+d_s^{(j)})\rangle=\mu\langle v^{(j)}+d_x^{(j)},v^{(j)}+d_s^{(j)}\rangle.$$

Using equation (22), it is obtained that

$$\begin{aligned}\langle v^{(j)}+d_x^{(j)},v^{(j)}+d_s^{(j)}\rangle&=\langle v^{(j)},v^{(j)}\rangle+\langle v^{(j)},d_x^{(j)}+d_s^{(j)}\rangle+\langle d_x^{(j)},d_s^{(j)}\rangle\\&=\langle v^{(j)},v^{(j)}\rangle+\langle v^{(j)},e^{(j)}-v^{(j)}\rangle+\langle d_x^{(j)},d_s^{(j)}\rangle\\&=\mathrm{Tr}(v^{(j)})+\mathrm{Tr}(d_x^{(j)}\circ d_s^{(j)}),\ j=1,2,\ldots,N.\end{aligned}$$

On the other hand, by the Cauchy-Schwartz inequality, Lemma 11 and the proof of Corollary 1, it follows that

$$\begin{aligned}\langle d_x^{(j)},d_s^{(j)}\rangle=\mathrm{Tr}(d_x^{(j)}\circ d_s^{(j)})&=\sum_{i=1}^{r_j}\lambda_i(d_x^{(j)}\circ d_s^{(j)})\\&\le\|e^{(j)}\|\|\lambda(d_x^{(j)}\circ d_s^{(j)})\|=\sqrt{r_j}\|d_x^{(j)}\circ d_s^{(j)}\|_F\\&<r_j\lambda_{\min}(v^{(j)})\le r_j(1+\sigma(v)).\end{aligned}$$

Thus, by the relations above and Lemma 11, it is obtained that

$$\begin{aligned}\langle x_+^{(j)},s_+^{(j)}\rangle&<\mu\left(\sum_{i=1}^{r_j}\lambda_i(v^{(j)})+r_j(1+\sigma(v))\right)\\&\le 2\mu r_j(1+\sigma(v))\\&<2\mu r_j\left(1+\frac{2}{1+\sqrt{3+4\kappa}}\right).\end{aligned}$$

Therefore,

$$\langle x_+, s_+\rangle = \sum_{j=1}^{N} \langle x_+^{(j)}, s_+^{(j)}\rangle < 2\mu r \left(\frac{3+\sqrt{3+4\kappa}}{1+\sqrt{3+4\kappa}}\right).$$

This completes the proof. ■

Define

$$v_+^{(j)} := \frac{P(w_+^{(j)})^{-\frac{1}{2}} x_+^{(j)}}{\sqrt{\mu}} \left[= \frac{P(w_+^{(j)})^{\frac{1}{2}} s_+^{(j)}}{\sqrt{\mu}}\right], \; j = 1, 2, \ldots, N, \tag{31}$$

where, $w_+^{(j)}$ is the scaling point of $x_+^{(j)}$ and $s_+^{(j)}$. Using Lemma 4, (27) and the second part of Lemma 3 respectively, it follows that

$$\begin{aligned} (v_+^{(j)})^2 &= \left(\frac{P(w_+^{(j)})^{\frac{1}{2}} s_+^{(j)}}{\sqrt{\mu}}\right)^2 \sim \frac{P(x_+^{(j)})^{\frac{1}{2}} s_+^{(j)}}{\mu} \\ &= \frac{\mu\Big(P\Big(P(w^{(j)})^{\frac{1}{2}}(v^{(j)}+d_x^{(j)})\Big)^{\frac{1}{2}} P(w^{(j)})^{-\frac{1}{2}}(v^{(j)}+d_s^{(j)})\Big)}{\mu} \\ &\sim P(v^{(j)} + d_x^{(j)})^{\frac{1}{2}}(v^{(j)} + d_s^{(j)}), \quad j = 1, 2, \ldots, N. \end{aligned} \tag{32}$$

Lemma 15 *The minimal eigenvalue of vector v_+ will always be greater than or equal to the square root of $1 - \sigma(v) - \frac{1+2\kappa}{2}\sigma(v)^2$.*

Proof: By using (32), Lemma 6, (28), Lemma 11 and Lemma 10, for $j = 1, 2, \ldots, N$, it can be deduced that

$$\begin{aligned} \lambda_{\min}\left((v_+^{(j)})^2\right) &= \lambda_{\min}\left(P(v^{(j)} + d_x^{(j)})^{\frac{1}{2}}(v^{(j)} + d_s^{(j)})\right) \\ &\geq \lambda_{\min}\left((v^{(j)} + d_x^{(j)}) \circ (v^{(j)} + d_s^{(j)})\right) \\ &= \lambda_{\min}\left(v^{(j)} + d_x^{(j)} \circ d_s^{(j)}\right) \\ &\geq \lambda_{\min}(v^{(j)}) - \|d_x^{(j)} \circ d_s^{(j)}\|_F \\ &\geq 1 - \sigma(v) - \|d_x^{(j)} \circ d_s^{(j)}\|_F \\ &\geq 1 - \sigma(v) - \|d_x \diamond d_s\|_F \\ &\geq 1 - \sigma(v) - \tfrac{1+2\kappa}{2}\sigma(v)^2, \end{aligned}$$

where the second inequality follows by Lemma 14 in [12]. Since for each $j = 1, 2, \ldots, N$, the above inequality is true, so

$$\lambda_{\min}(v_+) \geq \sqrt{1 - \sigma(v) - \frac{1+2\kappa}{2}\sigma(v)^2}.$$

This completes the proof. ■

The following theorem describes the effect of a μ-update and of a full-NT step on the proximity measure.

Theorem 3 *Let $\sigma(v) < \frac{2}{1+\sqrt{3+4\kappa}}$ and $\mu_+ = (1-\theta)\mu$, then*

$$\sigma(x_+, s_+; \mu_+) \le \frac{\theta\sqrt{r} + \sigma(v) + \frac{1+2\kappa}{2}\sigma(v)^2}{1-\theta+\sqrt{(1-\theta)(1-\sigma(v)-\frac{1+2\kappa}{2}\sigma(v)^2)}}.$$

Proof: Using equations (24), (31), (32), Lemma 5 and equation (28), it follows that

$$\begin{aligned}
\sigma(x_+, s_+; \mu_+)^2 &= \Big\|e - \frac{P(w^+)^{\frac{1}{2}}s^+}{\sqrt{\mu^+}}\Big\|_F^2 = \sum_{j=1}^N \Big\|e^{(j)} - \frac{P(w_+^{(j)})^{\frac{1}{2}}s_+^{(j)}}{\sqrt{\mu_+}}\Big\|_F^2 \\
&= \sum_{j=1}^N \Big\|e^{(j)} - \frac{v_+^{(j)}}{\sqrt{1-\theta}}\Big\|_F^2 = \sum_{j=1}^N \big\|(e^{(j)} - \frac{v_+^{(j)}}{\sqrt{1-\theta}}) \circ e^{(j)}\big\|_F^2 \\
&= \sum_{j=1}^N \big\|(e^{(j)} - (\frac{v_+^{(j)}}{\sqrt{1-\theta}})^2) \circ (e^{(j)} + \frac{v_+^{(j)}}{\sqrt{1-\theta}})^{-1}\big\|_F^2 \\
&\le \frac{1}{\Big(1+\lambda_{\min}(\frac{v_+}{\sqrt{1-\theta}})\Big)^2} \sum_{j=1}^N \big\|(e^{(j)} - (\frac{v_+^{(j)}}{\sqrt{1-\theta}})^2)\big\|_F^2 \\
&= \frac{1}{\Big(1+\lambda_{\min}(\frac{v_+}{\sqrt{1-\theta}})\Big)^2} \sum_{j=1}^N \Big\|e^{(j)} - \frac{P(v^{(j)}+d_x^{(j)})^{\frac{1}{2}}(v^{(j)}+d_s^{(j)})}{1-\theta}\Big\|_F^2 \\
&\le \frac{1}{\Big(1+\lambda_{\min}(\frac{v_+}{\sqrt{1-\theta}})\Big)^2} \sum_{j=1}^N \Big\|e^{(j)} - \frac{(v^{(j)}+d_x^{(j)}) \circ (v^{(j)}+d_s^{(j)})}{1-\theta}\Big\|_F^2 \\
&= \frac{1}{\Big(1+\lambda_{\min}(\frac{v_+}{\sqrt{1-\theta}})\Big)^2} \sum_{j=1}^N \Big\|e^{(j)} - \frac{v^{(j)}+d_x^{(j)} \circ d_s^{(j)}}{1-\theta}\Big\|_F^2 \\
&= \frac{1}{1-\theta}\frac{1}{\Big(\sqrt{1-\theta}+\lambda_{\min}(v_+)\Big)^2} \sum_{j=1}^N \big\|(1-\theta)e^{(j)} - v^{(j)} - d_x^{(j)} \circ d_s^{(j)}\big\|_F^2 \\
&= \frac{1}{1-\theta}\frac{1}{\Big(\sqrt{1-\theta}+\lambda_{\min}(v_+)\Big)^2} \big\|(1-\theta)e - v - d_x \diamond d_s\big\|_F^2.
\end{aligned}$$

Therefore, it can be deduced that

$$\begin{aligned}
\sigma(x_+, s_+; \mu_+) &\le \frac{1}{\sqrt{1-\theta}}\frac{1}{\sqrt{1-\theta}+\lambda_{\min}(v_+)}\big\|(1-\theta)e - v - d_x \diamond d_s\big\|_F \\
&\le \frac{1}{\sqrt{1-\theta}}\frac{\sigma(v)+\theta\sqrt{r}+\frac{1+2\kappa}{2}\sigma(v)^2}{\sqrt{1-\theta}+\sqrt{1-\sigma(v)-\frac{1+2\kappa}{2}\sigma(v)^2}},
\end{aligned}$$

where, the last inequality follows from Lemma 15, the triangle inequality, (24) and Lemma 10. This completes the proof. ∎

4.3 The choice of τ and θ

In this section a threshold τ and an update parameter θ are determined, so that at the start of the iteration, $\sigma(x, s; \mu) \leq \tau$. After the full-NT step and a μ-update, the property $\sigma(x_+, s_+; \mu_+) \leq \tau$ should be maintained. In this case, by Theorem 3, it suffices to have

$$\frac{\theta\sqrt{r} + \sigma(v) + \frac{1+2\kappa}{2}\sigma(v)^2}{1-\theta+\sqrt{(1-\theta)(1-\sigma(v)-\frac{1+2\kappa}{2}\sigma(v)^2)}} \leq \tau.$$

The left-hand side of the above inequality is monotonically increasing with respect to $\sigma(v)$, which implies that

$$\frac{\theta\sqrt{r} + \sigma(v) + \frac{1+2\kappa}{2}\sigma(v)^2}{1-\theta+\sqrt{(1-\theta)(1-\sigma(v)-\frac{1+2\kappa}{2}\sigma(v)^2)}} \leq \frac{\theta\sqrt{r} + \tau + \frac{1+2\kappa}{2}\tau^2}{1-\theta+\sqrt{(1-\theta)(1-\tau-\frac{1+2\kappa}{2}\tau^2)}}.$$

Thus, $\sigma(x_+, s_+; \mu_+) \leq \tau$ is satisfied if

$$\frac{\theta\sqrt{r} + \tau + \frac{1+2\kappa}{2}\tau^2}{1-\theta+\sqrt{(1-\theta)(1-\tau-\frac{1+2\kappa}{2}\tau^2)}} \leq \tau. \tag{33}$$

At this stage, if $\tau = \frac{1}{1+\sqrt{3+4\kappa}}$ and $\theta = \frac{1}{3\sqrt{6}(1+2\kappa)\sqrt{r}}$, the inequality (33) certainly holds. This means that $x, s \in \mathrm{int}\mathcal{K}$ and $\sigma(x, s; \mu) \leq \tau$ are maintained during the algorithm. Thus, the algorithm is well-defined.

4.4 Complexity bound

Lemma 16 *If the barrier parameter μ has the initial value μ^0 and is repeatedly multiplied by $1-\theta$, with $0 < \theta < 1$, then after at most*

$$\left\lceil \frac{1}{\theta} \log \frac{4r\mu^0}{\epsilon} \right\rceil$$

iterations, $\langle x, s\rangle \leq \epsilon$.

Proof: According to Lemma 14, after k iterations, the duality gap satisfies

$$\begin{aligned}\langle x^k, s^k\rangle &\leq 2r\Big(\frac{3+\sqrt{3+4\kappa}}{1+\sqrt{3+4\kappa}}\Big)(1-\theta)^k\mu^0\\ &\leq 4r(1-\theta)^k\mu^0.\end{aligned}$$

Then, the inequality $\langle x^k, s^k\rangle \leq \epsilon$ holds if

$$4r(1-\theta)^k\mu^0 \leq \epsilon.$$

Taking logarithms on both sides, it follows that

$$k \log(1-\theta) + \log\left(4r\mu^0\right) \leq \log \epsilon,$$

and using $\log(1-\theta) \leq -\theta, 0 < \theta < 1$, it is observed that the above inequality holds if

$$-k\theta + \log\left(4r\mu^0\right) \leq \log \epsilon.$$

This gives

$$k \geq \frac{1}{\theta} \log \frac{4r\mu^0}{\epsilon},$$

which completes the proof. ∎

Theorem 4 *Let $\theta = \frac{1}{3\sqrt{6}(1+2\kappa)\sqrt{r}}$ and $\tau = \frac{1}{1+\sqrt{3+4\kappa}}$, then the algorithm requires at most*

$$\mathcal{O}((1+2\kappa)\sqrt{r} \log \frac{r\mu^0}{\epsilon})$$

iterations. The output is a primal-dual pair (x, s) satisfying $\langle x, s\rangle \leq \epsilon$.

Proof: Let $\theta = \frac{1}{3\sqrt{6}(1+2\kappa)\sqrt{r}}$, by using Lemma 16, the proof is straightforward. ∎

5 Conclusion

A novel path-following interior-point algorithm is proposed for the Cartesian $P_*(\kappa)$-SCLCP and analyzed a full Nesterov-Todd step IPM based on the modified Nesterov-Todd direction. The complexity bound of the new algorithm, namely, $\mathcal{O}((1+2\kappa)\sqrt{r} \log \frac{r\mu^0}{\epsilon})$ was obtained. An interesting topic for further research may be the development of full modified Nesterov-Todd step into an infeasible case.

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