A finite source perishable inventory system with second optional service and server interruptions

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Abstract

In this article, a service facility inventory system with server interruptions and a finite number of sources are considered. The inventory is replenished according to \((s,S)\) ordering policy. Using the matrix methods, the stationary distribution of the stock level, server status and waiting area level is obtained in the steady state case. The Laplace-Stieltjes transform of the waiting time of the tagged customer is derived. Many impartment system performance measures are derived and the total expected cost rate is computed under a suitable cost structure. The results are illustrated numerically.

Key words: Essential and optional service, inventory with service time, service interruption, repair, finite source.

1 Introduction

Schwarz et al. [18] introduced the idea of inventory with positive service time. They have assumed that when the stock level is empty, each arriving customer enters into the queue. Berman et al. [2] derived deterministic approximations for a queueing-inventory system with a service facility. Berman & Kim [3] analyzed a single server inventory model with the assumption of instantaneous replenishment and a service facility. Berman & Sapna [5] considered a queueing-inventory model with single server and finite capacity. They assumed that customer arrival follows Poisson process, arbitrarily distributed service times and zero lead times. Berman & Kim [4] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a reorder policy that maximized the system proceeds.

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& Anbazhagan [13] studied a queueing-inventory system with $N$ policy and the finite waiting hall. They have assumed that if the customer level reaches a prefixed level $N$, then the server starts service immediately. Otherwise, server does not provide service to the waiting customers. Two other papers where a service facility inventory model is considered, are by Krishnamoorthy et al. [14, 15]. The paper [14] is analyzed for a queueing-inventory model with instantaneous replenishment and the service process is subject to interruptions. The discussion in [15] is a retrial queueing-inventory model with positive lead-time and the service process is subject to interruptions. Jeganathan & Periyasamy [11] studied a perishable inventory system with repeated attempts and the service process is subject to interruptions. They analyzed the system with the restriction that both the waiting area and orbit size are finite. Jeganathan et al. [9] discussed a service facility inventory system with multiple server vacations and server is subject to interruptions. For a detailed study in the service facility inventory models with an infinite number of sources the reader is referred to [1, 6, 8, 10, 12, 22].

An (s, S) inventory system with finite source was first initiated by Sivakumar [20]. He assumed that lifetime of each commodity is exponential and lead-time distribution is exponential. The author considered the constant retrial policy i.e., the probability of retrial is independent of the number of demands in the orbit. Shophia Lawrence et al. [19] considered a service facility inventory management system with the service time, the lead time are assumed to have Phase type distribution and finite source. Yadavalli et al. [23] analysed a service facility retrial inventory model with finite source and multi homogeneous servers. Yadavalli et al. [21] studied a two heterogeneous queueing-inventory system including one server is perfectly reliable and another server is subject to interruptions. In a recent paper, Jeganathan [7] analysed a mixed priority retrial inventory system with additional optional service and finite source.

A finite source queueing-inventory model with server interruptions and extra optional service is motivated by the service facility system with limited customers such as military canteen providing service to soldiers or a company canteen serving the members of the specific working area in the business. In this paper, a continuous review (s, S) inventory model with server interruptions, second optional service and finite source simultaneously are considered.

In the next section, the mathematical model is explained and the notations used in this paper are defined. Model analysis and the steady state analysis are proposed in section 3. In section 4, the waiting time analysis of a customers in the waiting area is discussed. Various important performance measures are derived in section 5. In section 6, the total expected cost rate is derived. In section 7, optimality of the cost function and its sensitivity with respect to various parameters using numerical examples are presented. The last section is meant for conclusion.

2 Model description

In this work, finite-source queueing-inventory models with the following assumption are studied. Consider a service facility wherein perishable items are stored and the items are distributed to the demanding customers. The maximum stock level is $S$. The customers
are generated by a finite number of identical sources \( N \), \((1 < N < \infty)\) and the demand time points form a quasi-random distribution with rate \( \lambda (> 0) \). An arriving customer finds the system either when the empty stock level or the server is busy or the server is on interruptions, then with probability \( r \) he/she enters into the waiting area. Otherwise, he/she balks (do not join) with probability \( 1 - r \). There is a single waiting area for the customers and the demand is for single item per customer. Before distributing items to the demanding customers, some primary service on the item is given first. In this article this type of service is referred to as first essential service (FES). This system has a single server who gives preliminary FES indicated by the rate \( \mu_1 \) to all arriving customers one by one according to FIFO (first in first out) discipline. When the FES of a customer is completed, the server may offer a second optional service (SOS) indicated by the rate \( \mu_2 \) with probability \( p \) to only those customers who opt for it otherwise leaves the system with the complementary probability \( q \), where \( p+q=1 \). While the server is in working state it may be interrupted at any time with interruption rate \( \alpha_1 \) during first essential service (FES) and \( \alpha_2 \) during second optional service (SOS). When the server interruption occurs, it is immediately sent for repairing where repair time indicated by \( \eta_1 \) for FES and \( \eta_2 \) for SOS. After repairing, the server provides residual service of the customers of both of the phases (FES or SOS). It is assumed that if the server is in interruption, no more interruption can be caused on the server. The service times of the FES and SOS, and the interruption times are assumed to follow an exponential distribution.

The operating policy is \((s, S)\) policy with exponential lead times for the ordered items. According to the ordering policy, when the stock level downfall to \( s \), an order for \( Q(=S-s>s+1) \) items are placed. Lifetime of each item has negative exponential distribution with rate \( \gamma > 0 \). The positive lead-time of the replenishment is assumed to be exponential with the rate \( \beta (> 0) \). All stochastic processes involved in the system are independent of each other. The notation used in this paper follows below.

\[ e \] : a column vector of appropriate dimension containing all ones
\[ 0 \] : zero matrix
\[ [A]_{ij} \] : entry at \((i,j)^{th}\) position of a matrix \( A \)
\[ \delta_{ij} \] : \[ \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \]
\[ \delta_{ij} \] : \[ 1 - \delta_{ij} \]
\[ H(x) : \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \]
\[ k \in V_i^j : k = i, i+1, \ldots, j \]
\[ \sum_{i=j}^k c_i : \begin{cases} c_j c_{j-1} \cdots c_k & \text{if } j \geq k \\ 1 & \text{if } j < k \end{cases} \]

3 Analysis

Let \( L(t) \) and \( X(t) \) respectively, denote the inventory level and the number of customers in the waiting area at time \( t \), \( L(t) \in \{0, 1, \ldots, S\} \) and \( X(t) \in \{0, 1, \ldots, N\} \) and let \( Y(t) \) denote
the status of the server, given by

\[
Y(t) = \begin{cases} 
0, & \text{if the server is idle at time } t, \\
1, & \text{if the server is providing first essential service to a customer at time } t, \\
2, & \text{if the server is providing second optional service to a customer at time } t, \\
3, & \text{if the server is on interruption during FES at time } t, \\
4, & \text{if the server is on interruption during SOS at time } t.
\end{cases}
\]

From the assumptions made on the input and output processes, it can be shown that the stochastic process \( I(t) = \{(L(t), Y(t), X(t)), t \geq 0\} \) is a continuous time Markov chain with discrete state space given by \( E = E_a \cup E_b \cup E_c \cup E_d \) where

\[
\begin{align*}
E_a & : \{(0,0,i_3) \mid i_3 = 0,1,2,\ldots,N,\} \\
E_b & : \{(i_1,0,0) \mid i_1 = 1,2,\ldots,S,\} \\
E_c & : \{(i_1,i_2,i_3) \mid i_1 = 1,2,\ldots,S, i_2 = 1,3, i_3 = 1,2,\ldots,N, \} \\
E_d & : \{(i_1,i_2,i_3) \mid i_1 = 0,1,2,\ldots,S, i_2 = 2,4, i_3 = 1,2,\ldots,N. \}
\end{align*}
\]

To determine the infinitesimal generator
\[
\Theta = \left\{ d((i_1,i_2,i_3),(j_1,j_2,j_3)) \right\}, \ (i_1,i_2,i_3),(j_1,j_2,j_3) \in E
\]

of this process the following arguments are used:

- Let \( Y(t) = 0; \)
  - a customer arrival in state \((0,0,i_3)\) will increases the number of customers in the waiting hall by one unit and the rate of the transition \( d((0,0,i_3),(0,0,i_3+1)) \) is given by \( r(N-i_3)\lambda, \ i_3 = 0,1,2,\ldots,N-1. \)
  - a transition from state \((i_1,0,0)\) to state \((i_1,1,1), i_1 = 1,2,\ldots,S, \) with intensity of transition \(N\lambda, \) when a customer arrives.
  - a transition from state \((i_1,0,0)\) to state \((i_1-1,0,0), \) will take place when any one of \(i_1\) items fails at a rate of \(\gamma; \) thus intensity for this transition is \(i_1\gamma, \ i_1 = 1,2,\ldots,S.\)

- Let \( Y(t) = 1 \) or \(3; \)
  - the arrival of the customer makes a transition from state \((i_1,i_2,i_3)\) to \((i_1,i_2,i_3+1)\) with intensity of transition \( r(N-i_3)\lambda, \ i_1 = 1,2,\ldots,S, \ i_2 = 1,3, i_3 = 1,2,\ldots,N-1. \)
  - when FES is completed, then with probability \(p\) the customer may ask for SOS, in this case his/her SOS will immediately commence. The system change from state \((i_1,1,i_3)\) to state \((i_1-1,2,i_3), \ i_1 = 1,2,\ldots,S, \ i_3 = 1,2,\ldots,N, \) with intensity of this transition \(p\mu_1\) or with probability \(q\) he/she may opt to leave the system, in this case both the customer level in the waiting hall and the inventory level decrease by one unit and also if the server finds both \(i_1 > 0\) and \(i_3 > 0, \) then server becomes busy, otherwise \(i.e.,\) if either \(i_3 = 0\) or \(i_1 = 0, \) then server becomes idle. The state of the system change from state \((1,1,i_3)\) to state \((0,0,i_3-1), \ i_3 = 1,2,\ldots,N, \) or \((i_1,1,1)\) to state \((i_1-1,0,0), i_1 = 2,\ldots,S, \) or \((i_1,1,i_3)\) to state \((i_1-1,1,i_3-1), i_1 = 2,\ldots,S, i_3 = 2,\ldots,N, \) with intensity of this transition \(q\mu_1.\)
Hence, we have

- a transition from state \((i_1, i_2, i_3)\) to state \((i_1 - 1, i_2, i_3)\), takes place when any one of the \((i_1 - 1)\) items of the commodity perishes at a rate \(\gamma\) and the intensity for this transition is \((i_1 - 1)\gamma\), where \(i_1 = 2, 3, \ldots, S, i_2 = 1, 3, i_3 = 1, 2, \ldots, N\).

- a transition from state \((i_1, 1, i_3)\) to state \((i_1, 3, i_3)\), \(i_1 = 1, 2, 3, \ldots, S, i_3 = 1, 2, \ldots, N\) will take place with intensity of transition \(\alpha_1\), when the interruption occurs during FES.

- a transition from state \((i_1, 3, i_3)\) to state \((i_1, 1, i_3)\), \(i_1 = 1, 2, 3, \ldots, S, i_3 = 1, 2, \ldots, N\) will take place with intensity of transition \(\eta_1\), when the repair completion during FES.

- Let \(Y(t)= 2\) or \(4\);

- the arrival of the customer makes a transition from state \((i_1, i_2, i_3)\) to \((i_1, i_2, i_3 + 1)\) with intensity of transition \(r(N - i_3)\lambda, i_1 = 0, 1, 2, \ldots, S, i_2 = 2, 4, i_3 = 1, 2, \ldots, N - 1\).

- the completion of optional service for a customer makes a transition from state \((0, 2, i_3)\) to state \((0, 0, i_3 - 1)\), \(i_3 = 1, 2, \ldots, N\), or from state \((i_1, 2, 1)\) to state \((i_1, 0, 0)\), \(i_1 = 1, 2, 3, \ldots, S\), or from state \((i_1, 2, i_3)\) to state \((i_1, 1, i_3 - 1)\), \(i_1 = 1, 2, 3, \ldots, S, i_3 = 2, \ldots, N\), with intensity of transition \(\mu_2\).

- a transition from state \((i_1, 2, i_3)\) to state \((i_1, 4, i_3)\), \(i_1 = 0, 1, 2, 3, \ldots, S, i_3 = 1, 2, \ldots, N\) will take place with intensity of transition \(\alpha_2\), when the interruption occurs during SOS.

- a transition from state \((i_1, 4, i_3)\) to state \((i_1, 2, i_3)\), \(i_1 = 0, 1, 2, 3, \ldots, S, i_3 = 1, 2, \ldots, N\) will take place with intensity of transition \(\eta_2\), when the repair completion during SOS.

- A passage from \((0, 0, 0)\) to \((Q, 0, 0)\), where \(Q(= S - s)\), or from \((0, 0, i_3)\) to \((Q, 1, i_3)\) for \(i_3 = 1, 2, \ldots, N\), or from \((0, i_2, i_3)\) to \((Q, i_2, i_3)\) for \(i_2 = 2, 4, i_3 = 1, 2, \ldots, N\), or from \((i_1, i_2, i_3)\) to \((i_1 + Q, i_2, i_3)\) for \(i_1 = 1, 2, \ldots, s\), \(i_2 = 1, 2, 3, 4, i_3 = 1, 2, \ldots, N\), or from \((i_1, 0, 0)\) to \((i_1 + Q, 0, 0)\) for \(i_1 = 1, 2, \ldots, s\), will take place with intensity of transition \(\beta\) when a replenishment for \(Q\) items occurs.

- For other transition from \((i_1, i_2, i_3)\) to \((j_1, j_2, j_3)\), except \((i_1, i_2, i_3) \neq (j_1, j_2, j_3)\), the rate is zero.

- Finally, the intensity of passage for the state \((i_1, i_2, i_3)\) is given by

$$
\frac{\sum_{(j_1, j_2, j_3) \neq (i_1, i_2, i_3)} d((i_1, i_2, i_3), (j_1, j_2, j_3))}{d((i_1, i_2, i_3), (j_1, j_2, j_3))} =
$$

Hence, we have \(d((i_1, i_2, i_3), (j_1, j_2, j_3)) =\)
\[ \begin{cases} \beta, & j_1 = Q, \quad j_2 = i_2, \quad j_3 = i_3, \\
& i_1 = 0, \quad i_2 = 0, \quad i_3 = 0, \\
& \text{or} \\
& j_1 = Q, \quad j_2 = 1, \quad j_3 = i_3, \\
& i_1 = 0, \quad i_2 = 0, \quad i_3 \in V_1^N, \\
& \text{or} \\
& j_1 = Q, \quad j_2 = i_2, \quad j_3 = i_3, \\
& i_1 = 0, \quad i_2 = 2, 4, \quad i_3 \in V_1^N, \\
& \text{or} \\
& j_1 = i_1 + Q, \quad j_2 = i_2, \quad j_3 = i_3, \\
& i_1 \in V_1^S, \quad i_2 = 0, \quad i_3 = 0, \\
& \text{or} \\
& j_1 = i_1 + Q, \quad j_2 = i_2, \quad j_3 = i_3, \\
& i_1 \in V_1^S, \quad i_2 \in V_1^N, \quad i_3 \in V_2^N, \\
& \text{or} \\
& j_1 = i_1 - 1, \quad j_2 = i_2, \quad j_3 = i_3, \\
& i_1 \in V_1^S, \quad i_2 = 2, 4, \quad i_3 \in V_1^N, \\
& \text{or} \\
& (i_1 - 1)^\gamma, \quad j_1 = i_1 - 1, \quad j_2 = i_2, \quad j_3 = i_3, \\
& i_1 \in V_2^S, \quad i_2 = 1, 3, \quad i_3 \in V_1^N, \\
& \text{or} \\
& p \mu_1, \quad j_1 = i_1 - 1, \quad j_2 = 2, \quad j_3 = i_3, \\
& i_1 \in V_1^S, \quad i_2 = 1, \quad i_3 \in V_1^N, \\
& \text{or} \\
& q \mu_1, \quad j_1 = i_1 - 1, \quad j_2 = 0, \quad j_3 = i_3 - 1, \\
& i_1 = 1, \quad i_2 = 1, \quad i_3 \in V_1^N, \\
& \text{or} \\
& N \lambda, \quad j_1 = i_1, \quad j_2 = 1, \quad j_3 = i_3 + 1, \\
& i_1 \in V_1^S, \quad i_2 = 0, \quad i_3 = 0, \\
& \text{or} \\
& r(N - i_3) \lambda, \quad j_1 = i_1, \quad j_2 = 0, \quad j_3 = i_3 + 1, \\
& i_1 = 0, \quad i_2 = 0, \quad i_3 \in V_0^{N-1}, \end{cases} \]
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\[ j_1 = 0, \quad j_2 = i_2, \quad j_3 = i_3 + 1, \]
\[ i_1 = 0, \quad i_2 = 2, 4, \quad i_3 \in V_1^{N-1}, \]

or

\[ j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3 + 1, \]
\[ i_1 \in V_1^S, \quad i_2 \in V_1^{4}, \quad i_3 \in V_1^{N-1}, \]

\[ \mu_2, \]
\[ j_1 = i_1, \quad j_2 = 0, \quad j_3 = i_3 - 1, \]
\[ i_1 = 0, \quad i_2 = 2, \quad i_3 \in V_1^N, \]

or

\[ j_1 = i_1, \quad j_2 = 0, \quad j_3 = 0, \]
\[ i_1 \in V_1^S, \quad i_2 = 2, \quad i_3 = 1, \]

or

\[ j_1 = i_1 - 1, \quad j_2 = 1, \quad j_3 = i_3 - 1, \]
\[ i_1 \in V_1^S, \quad i_2 = 2, \quad i_3 \in V_2^N, \]

\[ \alpha_1, \]
\[ j_1 = i_1, \quad j_2 = 3, \quad j_3 = i_3, \]
\[ i_1 \in V_1^S, \quad i_2 = 1, \quad i_3 \in V_1^N, \]

\[ \alpha_2, \]
\[ j_1 = i_1, \quad j_2 = 4, \quad j_3 = i_3, \]
\[ i_1 \in V_0^S, \quad i_2 = 2, \quad i_3 \in V_1^N, \]

\[ \eta_1, \]
\[ j_1 = i_1, \quad j_2 = 1, \quad j_3 = i_3, \]
\[ i_1 \in V_1^S, \quad i_2 = 3, \quad i_3 \in V_1^N, \]

\[ \eta_2, \]
\[ j_1 = i_1, \quad j_2 = 2, \quad j_3 = i_3, \]
\[ i_1 \in V_0^S, \quad i_2 = 4, \quad i_3 \in V_1^N, \]

\[ -(r(N - i_3)\lambda \tilde{\delta}_{i_3,N} + \beta), \]
\[ j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \]
\[ i_1 = 0, \quad i_2 = 0, \quad i_3 \in V_0^N, \]

\[ -(r(N - i_3)\lambda \tilde{\delta}_{i_3,N} + \beta + \delta_{i_2}2\mu_2 + \delta_{i_22\alpha_2} + \delta_{i_24\eta_2}), \]
\[ j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \]
\[ i_1 = 0, \quad i_2 = 2, 4, \quad i_3 \in V_1^N, \]

\[ -(N\lambda + H(s - i_1)\beta + i_1\gamma), \]
\[ j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \]
\[ i_1 \in V_1^S, \quad i_2 = 0, \quad i_3 = 0, \]

\[ -(r(N - i_3)\lambda \tilde{\delta}_{i_3,N} + H(s - i_1)\beta + \delta_{i_21\mu_1} + \delta_{i_22\mu_2} + \delta_{i_21\alpha_1} + \delta_{i_22\alpha_2} + \delta_{i_21}(i_1 - 1)\gamma + \delta_{i_22}\tilde{i}_1\gamma), \]
\[ j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \]
\[ i_1 \in V_1^S, \quad i_2 = 1, \quad i_3 \in V_1^N, \]

\[ -(r(N - i_3)\lambda \tilde{\delta}_{i_3,N} + H(s - i_1)\beta + \delta_{i_23}\eta_1 + \delta_{i_24}\eta_2 + \delta_{i_24}i_1\gamma + \delta_{i_23}(i_1 - 1)\gamma), \]
\[ j_1 = i_1, \quad j_2 = i_2, \quad j_3 = i_3, \]
\[ i_1 \in V_1^S, \quad i_2 = 3, 4, \quad i_3 \in V_1^N, \]

\[ 0, \quad \text{Otherwise.} \]
Define the following ordered sets:

\[
\langle i_1, i_2 \rangle = \begin{cases} 
\langle i_1, 0, i_3 \rangle, & i_1 = 0; i_3 = 0, 1, 2, \ldots, N; \\
\langle i_1, 0, i_3 \rangle, & i_1 = 1, 2, \ldots, S; i_3 = 0; \\
\langle i_1, i_2, i_3 \rangle, & i_1 = 1, 2, \ldots, S; i_2 = 1, 2, 3, 4; i_3 = 1, 2, \ldots, N; \\
\langle i_1, i_2, i_3 \rangle, & i_1 = 0; i_2 = 2, 4; i_3 = 1, 2, \ldots, N;
\end{cases}
\]

\[
\langle i_1 \rangle = \begin{cases} 
\langle i_1, 0 \rangle, & i_1 = 0, 1, 2, \ldots S; \\
\langle i_1, i_2 \rangle, & i_1 = 1, 2, \ldots, S; i_2 = 1, 2, 3, 4; \\
\langle i_1, i_2 \rangle, & i_1 = 0; i_2 = 2, 4;
\end{cases}
\]

By ordering the sets of state space as \(\langle \langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \ldots, \langle S \rangle \rangle\), the infinitesimal generator \(\Theta\) can be conveniently expressed in a block partitioned matrix with entries

\[
\Theta_{i_1j_1} = \begin{cases} 
Z_{i_1} & j_1 = i_1, i_1 = 0, 1, 2, \ldots, S, \\
W_{i_1} & j_1 = i_1 - 1, i_1 = 1, 2, \ldots, S - 1, S, \\
C & j_1 = i_1 + Q, i_1 = 1, 2, \ldots, s, \\
C_1 & j_1 = i_1 + Q, i_1 = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

More explicitly,

\[
\Theta = \begin{pmatrix}
\begin{array}{cccc}
Z_S & W_S & W_{S-1} & \cdots \\
Z_{S-1} & W_{S-1} & \cdots & \\
\vdots & \vdots & \ddots & \\
Z_1 & W_1 & \cdots & \cdots Z_1 \\
0 & \cdots & \cdots & Z_0
\end{array}
\end{pmatrix}
\]

where

\[
[C_1]_{i_2j_2} = \begin{cases} 
C_0 & j_2 = i_2, i_2 = 0, \\
C_2 & j_2 = 1, i_2 = 0, \\
C_3 & j_2 = i_2, i_2 = 2, 4, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
[C_0]_{i_3j_3} = \begin{cases} 
\beta & j_3 = 0, i_3 = 0, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
[C_2]_{i_3j_3} = \begin{cases} 
\beta & j_3 = i_3, i_3 \in V_1^N \\
0, & \text{otherwise.}
\end{cases}
\]

\[
[C]_{i_2j_2} = \begin{cases} 
C_4 & j_2 = i_2, i_2 = 0, \\
C_3 & j_2 = i_2, i_2 = 1, 2, 3, 4, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
[C_4]_{i_3j_3} = \begin{cases} 
\beta & j_3 = 0, i_3 = 0, \\
0, & \text{otherwise.}
\end{cases}
\]
\[ [W_1]_{i_2 j_2} = \begin{cases} \begin{align*} E_0 & \quad j_2 = 2, \quad i_2 = 1, \\
D_0 & \quad j_2 = 0, \quad i_2 = 1, \\
F_1 & \quad j_2 = 2, \quad i_2 = 2, 4, \\
B_{00}^{(1)} & \quad j_2 = i_2, \quad i_2 = 0, \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [B_{00}^{(1)}]_{i_3 j_3} = \begin{cases} \begin{align*} \gamma & \quad j_3 = i_3, \quad i_3 = 0, \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [E_0]_{i_3 j_3} = \begin{cases} \begin{align*} p \mu_1 & \quad j_3 = i_3, \quad i_3 \in V_1^N \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [D_0]_{i_3 j_3} = \begin{cases} \begin{align*} q \mu_1 & \quad j_3 = i_3 - 1, \quad i_3 \in V_1^N \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [F_1]_{i_3 j_3} = \begin{cases} \begin{align*} \gamma & \quad j_3 = i_3, \quad i_3 \in V_1^N \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

For \( i_1 = 2, 3, \ldots, S \)

\[ [W_{i_1}]_{i_2 j_2} = \begin{cases} \begin{align*} E_0 & \quad j_2 = 2, \quad i_2 = 1, \\
F_0 & \quad j_2 = 0, \quad i_2 = 1, \\
E_{i_1} & \quad j_2 = 0, \quad i_2 = 0, \\
D_{(i_1 - 1)} & \quad j_2 = 1, \quad i_2 = 1 \\
F_{i_1} & \quad j_2 = i_2, \quad i_2 = 2, 4 \\
G_{(i_1 - 1)} & \quad j_2 = i_2, \quad i_2 = 3 \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [F_0]_{i_3 j_3} = \begin{cases} \begin{align*} q \mu_1 & \quad j_3 = 0, \quad i_3 = 1 \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [E_{i_1}]_{i_3 j_3} = \begin{cases} \begin{align*} i_1 \gamma & \quad j_3 = i_3, \quad i_3 = 0 \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [D_{(i_1 - 1)}]_{i_3 j_3} = \begin{cases} \begin{align*} (i_1 - 1) \gamma & \quad j_3 = i_3, \quad i_3 \in V_1^N \\
q \mu_1 & \quad j_3 = i_3 - 1, \quad i_3 \in V_2^N \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [F_{i_1}]_{i_3 j_3} = \begin{cases} \begin{align*} i_1 \gamma & \quad j_3 = i_3, \quad i_3 \in V_1^N \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]

\[ [Z_0]_{i_2 j_2} = \begin{cases} \begin{align*} V_0 & \quad j_2 = i_2, \quad i_2 = 0, \\
V_1 & \quad j_2 = i_2, \quad i_2 = 2, \\
V_2 & \quad j_2 = 4, \quad i_2 = 4, \\
R & \quad j_2 = 2, \quad i_2 = 4 \\
T_2 & \quad j_2 = 4, \quad i_2 = 2 \\
U & \quad j_2 = 0, \quad i_2 = 2 \\
0 & \quad \text{otherwise}. \end{align*} \end{cases} \]
\[
\begin{align*}
[U]_{i_3,j_3} &= \begin{cases} 
\mu_3, & j_3 = i_3 - 1, \quad i_3 \in V_1^N \\
0, & \text{otherwise}.
\end{cases} \\
[T_2]_{i_3,j_3} &= \begin{cases} 
\alpha_2, & j_3 = i_3, \quad i_3 \in V_1^N \\
0, & \text{otherwise}.
\end{cases} \\
[V_0]_{i_3,j_3} &= \begin{cases} 
(r(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-1} \\
-(r(N - i_3)\lambda\delta_{i_3N} + \beta), & j_3 = i_3, \quad i_3 \in V_0^N, \\
0, & \text{otherwise}.
\end{cases} \\
[V_1]_{i_3,j_3} &= \begin{cases} 
(r(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1} \\
-(r(N - i_3)\lambda\delta_{i_3N} + \beta + \mu_2 + \alpha_2), & j_3 = i_3, \quad i_3 \in V_1^N, \\
0, & \text{otherwise}.
\end{cases} \\
[V_2]_{i_3,j_3} &= \begin{cases} 
(r(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1} \\
-(r(N - i_3)\lambda\delta_{i_3N} + \beta + \eta_2), & j_3 = i_3, \quad i_3 \in V_1^N, \\
0, & \text{otherwise}.
\end{cases} \\
[H]_{i_3,j_3} &= \begin{cases} 
N\lambda, & j_3 = 1, \quad i_3 = 0, \\
0, & \text{otherwise}.
\end{cases} \\
[H_{i_1}]_{i_3,j_3} &= \begin{cases} 
-(N\lambda + i_1\gamma + H(s - i_1)\beta), & j_3 = i_3, \quad i_3 = 0, \\
0, & \text{otherwise}.
\end{cases} \\
[J]_{i_3,j_3} &= \begin{cases} 
\mu_2, & j_3 = 0, \quad i_3 = 1, \\
0, & \text{otherwise}.
\end{cases} \\
[J_0]_{i_3,j_3} &= \begin{cases} 
\mu_2, & j_3 = i_3 - 1, \quad i_3 \in V_2^N, \\
0, & \text{otherwise}.
\end{cases}
\end{align*}
\]

For \( i_1 = 1, 2, 3, \ldots, S \)

\[
\begin{align*}
[Z_{i_1}]_{i_2,j_2} &= \begin{cases} 
H & j_2 = 1, \quad i_2 = 0, \\
H_{i_1} & j_2 = i_2, \quad i_2 = 0, \\
J & j_2 = 0, \quad i_2 = 2, \\
J_0 & j_2 = 1, \quad i_2 = 2, \\
J_{i_1} & j_2 = i_2, \quad i_2 = 1, \\
K_{i_1} & j_2 = 2, \quad i_2 = 2, \\
L_0 & j_2 = 1, \quad i_2 = 3, \\
L_{i_1} & j_2 = i_2, \quad i_2 = 3, \\
R & j_2 = 2, \quad i_2 = 4, \\
R_{i_1} & j_2 = 2, \quad i_2 = 4, \\
T_1 & j_2 = 3, \quad i_2 = 1, \\
T_2 & j_2 = 4, \quad i_2 = 2, \\
0, & \text{otherwise}.
\end{cases} \\
[J_{i_1}]_{i_3,j_3} &= \begin{cases} 
(r(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\
-(r(N - i_3)\lambda\delta_{i_3N} + \mu_1 + H(s - i_1)\beta + (i_1 - 1)\gamma + \alpha_1), & j_3 = i_3, \quad i_3 \in V_1^N, \\
0, & \text{otherwise}.
\end{cases} \\
[K_{i_1}]_{i_3,j_3} &= \begin{cases} 
(r(N - i_3)\lambda, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\
-(r(N - i_3)\lambda\delta_{i_3N} + \mu_2 + H(s - i_1)\beta + i_1\gamma + \alpha_2), & j_3 = i_3, \quad i_3 \in V_1^N, \\
0, & \text{otherwise}.
\end{cases} \\
[L_0]_{i_3,j_3} &= \begin{cases} 
\eta_1, & j_3 = i_3, \quad i_3 \in V_1^N, \\
0, & \text{otherwise}.
\end{cases}
\end{align*}
\]
3.1 Steady state analysis

It can be noted that $Z_{i_1}, W_{i_1}, i_1 = 2, \ldots, S, Z_1,$ and $C$ are square matrices of order $(4N + 1)$. $C_1$ is of size $(3N + 1) \times (4N + 1)$. $Z_0$ is a square matrices of order $(3N + 1)$, $W_1$ is of size $(4N + 1) \times (3N + 1)$. The sub matrices $C_3, E_0, F_1, T_1, T_2, R, J_0, L_0, R_1, K_1, J_1, L_1, D_{i_1-1}, F_{i_1}, G_{i_1-1}, R_{i_1}, K_{i_1}, J_{i_1}, L_{i_1}, i_1 = 2, 3, \ldots, S,$ are square matrices of order $N$. $C_4, H_1, E_{i_1}, H_{i_1}, i_1 = 2, 3, \ldots, S,$ are square matrices of order 1. $C_0$ and $J$ are matrices of order $(N + 1) \times 1$. $D_0, C_2, B_{00}, F_0$ and $H$ are matrices of order $N \times (N + 1)$, $(N + 1) \times N$, $1 \times (N + 1)$, $N \times 1$ and $1 \times N$ respectively.

The computation of steady state probability vector $\Phi = (\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(S)})$, by solving the following set of equations,

\[
\Phi^{i_1} W_{i_1} + \Phi^{i_1-1} Z_{i_1-1} = 0, \quad i_1 = 1, 2, \ldots, Q, \\
\Phi^{i_1} W_{i_1} + \Phi^{i_1-1} Z_{i_1-1} + \Phi^{i_1-1-Q} C_1 = 0, \quad i_1 = Q + 1, \\
\Phi^{i_1} W_{i_1} + \Phi^{i_1-1} Z_{i_1-1} + \Phi^{i_1-1-Q} C = 0, \quad i_1 = Q + 2, Q + 3, \ldots, S, \\
\Phi^S Z_S + \Phi^S C = 0.
\]
subject to conditions $\Phi \Theta = 0$ and $\sum \sum_{(i_1, i_2, i_3)} \phi^{(i_1, i_2, i_3)} = 1$.

This is done by the following algorithm.

**Step 1.** Solve the following system of equations to find the value of $\Phi^Q$

$$
\phi^Q \left[ \left\{ \begin{array}{l}
(-1)^Q \sum_{j=0}^{s-1} \left[ \left( s+1-j \atop k=Q \right) W_k Z_{k-1} \right] C Z_{s-j} \left( \frac{Q+2}{l=S-j} W_l Z_{l-1} \right) \right] \right] W_{Q+1} + Z_Q + \left\{ (-1)^Q \sum_{j=Q}^{1} W_j Z_{j-1} \right\} C = 0,
\right.
$$

and

$$
\phi^Q \left[ \sum_{i_1=0}^{Q-1} \left( (-1)^{Q-i_1} \sum_{j=Q}^{Q-i_1+1} W_j Z_{j-1} \right) + I 
+ \sum_{i_1=Q+1}^{S} \left( (-1)^{2Q-i_1+1} \sum_{j=Q}^{S-i_1} \left[ \left( s+1-j \atop k=Q \right) W_k Z_{k-1} \right] C Z_{s-j} \left( \frac{i_1+1}{l=S-j} W_l Z_{l-1} \right) \right) \right] \pi = 1.
$$

**Step 2.** Compute the values of $\Omega_{i_1} = (-1)^{Q-i} \phi^Q \sum_{j=Q}^{Q-i+1} W_j Z_{j-1}$, $i_1 = Q-1, Q-2, \ldots, 0$

$$
= (-1)^{2Q-i_1+1} \phi^Q \sum_{j=Q}^{S-i_1} \left[ \left( s+1-j \atop k=Q \right) W_k Z_{k-1} \right] C Z_{s-j} \left( \frac{i_1+1}{l=S-j} W_l Z_{l-1} \right),
$$

$$
= I,
$$

$i_1 = S, S-1, \ldots, Q + 1$

$i_1 = Q$

**Step 3.** Using, Step 1 and Step 2, calculate the value of $\Phi^{(i_1)}$, $i_1 = 0, 1, \ldots, S$. That is,

$$
\Phi^{(i_1)} = \phi^{(Q)} \Omega_{i_1}, \quad i_1 = 0, 1, \ldots, S.
$$

### 3.2 Waiting time analysis

In this section, the aim is to derive the waiting time for the customer. The specific is the time between the arrival times of the customer and immediate upon which he gets service. We will symbolize this continuous time random variable as $W$. The aim is to derive the probability distribution of $W$ and to derive $n^{th}$ order moments of $W$. Note that $W$ is zero when the is in the state $(i_1, 0, 0)$, $i_1 \in V_1^S$. Consequently, the probability that the customer does not have to wait is given by

$$
P\{W = 0\} = \sum_{i_1=1}^{S} \phi^{(i_1, 0, 0)}.
$$

To obtain the distribution of $W$, some auxiliary variables are defined. Let us consider the Markov process at an arbitrary time $t$ and assume that the system in the
state \((i_1, i_2, i_3), i_3 > 0\). We tag any of those waiting customer and \(W_{(i_1, i_2, i_3)}\) denotes the time until the selected customer gets the desired item. Let \(W^*(y) = E[e^{-yW}]\) and \(W^*_{(i_1, i_2, i_3)}(y) = E[e^{-yW_{(i_1, i_2, i_3)}}]\) respectively, denote the unconditional and conditional waiting time. Then, we have

\[
W^*(y) = \sum_{i_1=1}^{S} \phi^{(i_1,0,0)} + \sum_{i_3=0}^{N-1} \phi^{(0,0,i_3)} W^*_{(0,0,i_3+1)}(y) \\
+ \sum_{i_1=1}^{S} \sum_{i_3=1}^{N-1} (\phi^{(i_1,1,i_3)} W^*_{(i_1,1,i_3+1)}(y) + \phi^{(i_1,3,i_3)} W^*_{(i_1,3,i_3+1)}(y)) \\
+ \sum_{i_1=1}^{S} \sum_{i_3=1}^{N-1} (\phi^{(i_1,2,i_3)} W^*_{(i_1,2,i_3+1)}(y) + \phi^{(i_1,4,i_3)} W^*_{(i_1,4,i_3+1)}(y)).
\]

(1)

To derive \(W^*_{(i_1, i_2, i_3)}\), we introduce an auxiliary Markov chain on the state space \(E^* = E_a \cup E_c \cup E_d \cup \{\ast\}\), where \(\{\ast\}\) represents an absorbing state. The chain is on a state \((i_1, i_2, i_3)\), we apply a first-step argument in the auxiliary chain to resolve \(W^*_{(i_1, i_2, i_3)}(y)\). Then (see [16], Theorem 6.21) the functions \(W^*_{(i_1, i_2, i_3)}(y)\), \((i_1, i_2, i_3) \in E\) are the smallest non-negative solution to the system

For \(i_1 = 0, \quad i_2 = 0, \quad 1 \leq i_3 \leq N,\)

\[
w_1 W^*_{(0,0,i_3)}(y) - r(N - i_3)\lambda \bar{\delta}_{i_2, N} W^*_{(0,0,i_3+1)}(y) - \beta \bar{\delta}_{i_3} W^*_{(Q,0,0)}(y) - \beta \bar{\delta}_{i_3} W^*_{(Q,1,i_3)}(y) = 0
\]

(2)

where

\[
w_1 = y + r(N - i_3)\lambda \bar{\delta}_{i_2, N} + \beta \bar{\delta}_{i_3} + \beta \bar{\delta}_{i_3}
\]

For \(1 \leq i_1 \leq S, \quad i_2 = 1, 3, \quad 1 \leq i_3 \leq N\)

\[
w_2 W^*_{(i_1, i_2, i_3)}(y) - r(N - i_3)\lambda \delta_{i_3} N W^*_{(i_1, i_2, i_3+1)}(y) - (i_1 - 1)\gamma \delta_{i_1} W^*_{(i_1-1, i_2, i_3)}(y) \\
- \beta H(s - i_1) W^*_{(i_1+Q,i_2,i_3)}(y) - p\mu_1 \delta_{i_1} \delta_{i_2} W^*_{(i_1-1,2,i_3-1)}(y) - q\mu_1 \delta_{i_1} \delta_{i_2} W^*_{(i_1-1,0,i_3-1)}(y) \\
- p\mu_1 \delta_{i_1} \delta_{i_2} \delta_{i_3} W^*_{(i_1-1,2,i_3-2)}(y) - q\mu_1 \delta_{i_1} \delta_{i_2} \delta_{i_3} W^*_{(i_1-1,0,i_3-2)}(y) \\
- q\mu_1 \delta_{i_1} \delta_{i_2} \delta_{i_3} W^*_{(i_1-1,2,i_3-3)}(y) - \alpha_1 \delta_{i_2} W^*_{(i_1,3,i_3)}(y) - \eta_1 \delta_{i_2} W^*_{(i_1,4,i_3)}(y) = 0
\]

(3)

where

\[
w_2 = y + r(N - i_3)\lambda \bar{\delta}_{i_3} N + (i_1 - 1)\gamma \delta_{i_1} + \beta H(s - i_1) + p\mu_1 \delta_{i_1} \delta_{i_2} + q\mu_1 \delta_{i_1} \delta_{i_2} \\
+ p\mu_1 \delta_{i_1} \delta_{i_2} \delta_{i_3} + q\mu_1 \delta_{i_1} \delta_{i_2} \delta_{i_3} + p\mu_1 \delta_{i_1} \delta_{i_2} \delta_{i_3} + q\mu_1 \delta_{i_1} \delta_{i_2} \delta_{i_3} + \alpha_1 \delta_{i_2} + \eta_1 \delta_{i_2}
\]

For \(0 \leq i_1 \leq S, \quad i_2 = 2, 4, \quad 1 \leq i_3 \leq N\)

\[
w_3 W^*_{(i_1, i_2, i_3)}(y) - r(N - i_3)\lambda \bar{\delta}_{i_3} N W^*_{(i_1, i_2, i_3+1)}(y) - \beta H(s - i_1) W^*_{(i_1+Q,i_2,i_3)}(y) \\
- i_1 \gamma W^*_{(i_1-1, i_2, i_3)}(y) - \mu_2 \delta_{i_1} \delta_{i_2} W^*_{(i_1,0,i_3-1)}(y) - \mu_2 \delta_{i_1} \delta_{i_2} \delta_{i_3} W^*_{(i_1,0,i_3)}(y) \\
- \mu_2 \delta_{i_1} \delta_{i_2} \delta_{i_3} W^*_{(i_1,1,i_3-1)}(y) - \alpha_2 \delta_{i_2} W^*_{(i_1,4,i_3)}(y) - \eta_2 \delta_{i_2} W^*_{(i_1,4,i_3)}(y) = 0
\]

(4)

where

\[
w_3 = y + r(N - i_3)\lambda \bar{\delta}_{i_3} N + \beta H(s - i_1) + i_1 \gamma + \mu_2 \delta_{i_1} \delta_{i_2} \\
+ \mu_2 \delta_{i_1} \delta_{i_2} \delta_{i_3} + \mu_2 \delta_{i_1} \delta_{i_2} \delta_{i_3} + \alpha_2 \delta_{i_2} + \eta_2 \delta_{i_2}.
\]
Using the linear equations (2)–(4), we can compute the values of $W^*(y)$ for a given $y$ and also we can utilize the system of linear equations to obtain a recursive algorithm for calculating moments for the waiting times. By differentiating $(n+1)$ times (2)–(4) the system of linear equations, and evaluating at $y = 0$, we arrive at

For $i_1 = 0, \quad i_2 = 0, \quad 1 \leq i_3 \leq N$

$$w_4 E \left[ W_{(0,0,i_3)}^{(n+1)} \right] - r(N - i_3) \lambda \bar{\delta}_{i_3N} E \left[ W_{(0,0,i_3+1)}^{(n+1)} \right] - \beta \delta_{i_30} E \left[ W_{(Q,0,0)}^{(n+1)} \right]$$

$$- \beta \delta_{i_30} E \left[ W_{(Q,1,i_3)}^{(n+1)} \right] = (n+1)E \left[ W_{(0,0,i_3)}^{(n)} \right]$$

where

$$w_4 = r(N - i_3) \lambda \bar{\delta}_{i_3N} + \beta \delta_{i_30} + \beta \delta_{i_30}$$

For $1 \leq i_1 \leq S \quad i_2 = 1, 3, \quad 1 \leq i_3 \leq N$

$$w_5 E \left[ W_{(i_1,i_2,i_3)}^{(n+1)} \right] - r(N - i_3) \lambda \bar{\delta}_{i_3N} E \left[ W_{(i_1,i_2,i_3+1)}^{(n+1)} \right]$$

$$- (i_1 - 1) \gamma_i \delta_{i_11} E \left[ W_{(i_1-1,i_2,i_3)}^{(n+1)} \right] - \beta H(s - i_1) E \left[ W_{(i_1+Q,i_2,i_3)}^{(n+1)} \right]$$

$$- p \mu_1 \delta_{i_11} \delta_{i_21} E \left[ W_{(i_1-1,2,i_3-1)}^{(n+1)} \right] - q \mu_1 \delta_{i_11} \delta_{i_21} E \left[ W_{(i_1-1,0,i_3-1)}^{(n+1)} \right]$$

$$- p \mu_1 \delta_{i_11} \delta_{i_21} \delta_{i_31} E \left[ W_{(i_1-1,1,2,i_3)}^{(n+1)} \right] - q \mu_1 \delta_{i_11} \delta_{i_21} \delta_{i_31} E \left[ W_{(i_1-1,1,0,i_3)}^{(n+1)} \right]$$

$$- p \mu_1 \delta_{i_11} \delta_{i_21} \delta_{i_31} \delta_{i_41} E \left[ W_{(i_1-1,1,2,i_3)}^{(n+1)} \right] - q \mu_1 \delta_{i_11} \delta_{i_21} \delta_{i_31} \delta_{i_41} E \left[ W_{(i_1-1,1,1,i_3)}^{(n+1)} \right]$$

$$- \alpha \delta_{i_21} E \left[ W_{(i_1,1,3,i_3)}^{(n+1)} \right] - \eta \delta_{i_23} E \left[ W_{(i_1,1,1,i_3)}^{(n+1)} \right] = \eta \delta_{i_23} E \left[ W_{(i_1,1,1,i_3)}^{(n+1)} \right]$$

where

$$w_5 = r(N - i_3) \lambda \bar{\delta}_{i_3N} + (i_1 - 1) \gamma_i \delta_{i_11} + \beta H(s - i_1) + p \mu_1 \delta_{i_11} \delta_{i_21} + q \mu_1 \delta_{i_11} \delta_{i_21}$$

$$+ p \mu_1 \delta_{i_11} \delta_{i_21} \delta_{i_31} + q \mu_1 \delta_{i_11} \delta_{i_21} \delta_{i_31} + p \mu_1 \delta_{i_11} \delta_{i_21} \delta_{i_31} + q \mu_1 \delta_{i_11} \delta_{i_21} \delta_{i_31} + \alpha \delta_{i_21} + \eta \delta_{i_23}$$

For $0 \leq i_1 \leq S, \quad i_2 = 2, 4, \quad 1 \leq i_3 \leq N$

$$w_6 E \left[ W_{(i_1,i_2,i_3)}^{(n+1)} \right] - r(N - i_3) \lambda \bar{\delta}_{i_3N} E \left[ W_{(i_1+1,i_2,i_3+1)}^{(n+1)} \right] - \beta H(s - i_1) E \left[ W_{(i_1+Q,i_2,i_3)}^{(n+1)} \right]$$

$$- \mu \delta_{i_10} \delta_{i_22} E \left[ W_{(i_1,0,i_3-1)}^{(n+1)} \right] - \mu \delta_{i_10} \delta_{i_22} E \left[ W_{(i_1,0,i_3-1)}^{(n+1)} \right]$$

$$- \mu \delta_{i_10} \delta_{i_22} \delta_{i_31} E \left[ W_{(i_1,1,i_3-1)}^{(n+1)} \right] - \mu \delta_{i_10} \delta_{i_22} \delta_{i_31} E \left[ W_{(i_1,1,i_3-1)}^{(n+1)} \right]$$

$$- \mu \delta_{i_10} \delta_{i_22} \delta_{i_31} \delta_{i_41} E \left[ W_{(i_1,1,1,i_3)}^{(n+1)} \right] - \mu \delta_{i_10} \delta_{i_22} \delta_{i_31} \delta_{i_41} E \left[ W_{(i_1,1,1,i_3)}^{(n+1)} \right]$$

$$= (n+1)E \left[ W_{(i_1,i_2,i_3)}^{(n+1)} \right]$$

where

$$w_6 = r(N - i_3) \lambda \bar{\delta}_{i_3N} + \beta H(s - i_1) + \mu \delta_{i_10} \delta_{i_22} + \mu \delta_{i_10} \delta_{i_22} \delta_{i_31} + \mu \delta_{i_10} \delta_{i_22} \delta_{i_31} + \alpha \delta_{i_22} + \eta \delta_{i_24}$$
Equations (5)–(7) are used to determine the unknowns \( E \left[ W^{(n+1)}_{(i_1, i_2, i_3)} \right], (i_1, i_2, i_3) \in E \) in terms of the moments of one order less. Noticing that \( E \left[ W^{(n)}_{(i_1, i_2, i_3)} \right] = 1, \) for \( n = 0 \), we can obtain the moments up to a desired order in a recursive way.

For determine the moments of \( W \) we differentiate \( W^*(y) \) and evaluate at \( y = 0 \), we have

\[
E[W^{(n)}] = \delta_{0n} + \left[ \sum_{i_3=0}^{N-1} \phi^{(i_3)}_{0,0,0} E \left[ W^{(n)}_{(0,0,i_3)} \right] + \sum_{i_1=1}^{S} \sum_{i_3=1}^{N-1} \sum_{i_2=1}^{N} \phi^{(i_1, i_2, i_3)} E \left[ W^{(n)}_{(i_1, i_2, i_3+1)} \right] \\
+ \phi^{(i_1, i_2, i_3)} E \left[ W^{(n)}_{(i_1, i_2, i_3+1)} \right] \right) (1 - \delta_{0n})
\]

which provides the \( n^{th} \) moments of the unconditional waiting time in terms of conditional moments of the same order.

4 System performance measures

In this section, some measures of system performance in the steady state are derived. Using this, the total expected cost rate is derived.

4.1 Expected inventory level

Let \( \eta_I \) denote the excepted inventory level in the steady state, then

\[
\eta_I = \sum_{i_1=1}^{S} i_1 \phi^{(i_1, 0, 0)} + \sum_{i_1=1}^{S} \sum_{i_2=1}^{4} \sum_{i_3=1}^{N} i_1 \phi^{(i_1, i_2, i_3)}.
\]

4.2 Expected reorder rate

Let \( \eta_R \) denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from \( s + 1 \) to \( s \). This may occur in the following three cases:

- The server completes a first essential service for the customer.
- Any one of the \( s \) items fails when the server is busy/interruption during FES.
- Any one of the \( (s + 1) \) items fails when the server is idle/busy/interruption during SOS.

Hence, we get

\[
\eta_R = \sum_{i_3=1}^{N} \mu_1 \phi^{(s+1,1,i_3)} + (s + 1) \gamma \phi^{(s+1,0,0)} \\
+ \sum_{i_3=1}^{N} (s + 1) \gamma (\phi^{(s+1,2,i_3)} + \phi^{(s+1,4,i_3)}) + \sum_{i_3=1}^{N} s \gamma (\phi^{(s+1,1,i_3)} + \phi^{(s+1,3,i_3)}).
\]
4.3 Expected perishable rate

Let $\eta_P$ denote the expected perishable rate in the steady state, then

$$
\eta_P = \sum_{i_1=1}^{S} i_1 \gamma \phi^{(s+1,0,0)} + \sum_{i_1=2}^{N} \sum_{i_3=1}^{S} (i_1 - 1) \gamma \left( \phi^{(i_1,1,i_3)} + \phi^{(i_1,3,i_3)} \right)
+ \sum_{i_3=1}^{N} \sum_{i_1=1}^{S} i_1 \gamma \left( \phi^{(i_1,2,i_3)} + \phi^{(i_1,4,i_3)} \right).
$$

4.4 Expected number of customers in the waiting area

Let $\Gamma_1$ denote the expected number of customers in the steady state, then

$$
\Gamma_1 = \sum_{i_3=1}^{N} i_3 \phi^{(0,0,i_3)} + \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} i_3 \left( \phi^{(i_1,1,i_3)} + \phi^{(i_1,3,i_3)} \right)
+ \sum_{i_3=1}^{N} \sum_{i_1=0}^{S} i_3 \left( \phi^{(i_1,2,i_3)} + \phi^{(i_1,4,i_3)} \right).
$$

4.5 Expected waiting time

Let $\eta_W$ denote the expected waiting time of the customers in the waiting area. Then by Little's formula

$$
\eta_W = \frac{\Gamma_1}{\Gamma_2},
$$

where $\Gamma_1$ is the expected number of customers in the waiting area and the effective arrival rate of the customer [17], $\Gamma_2$ is given by

$$
\Gamma_2 = \sum_{i_3=0}^{N-1} r(N - i_3) \lambda \phi^{(0,0,i_3)} + \sum_{i_1=1}^{S} (N - i_3) \lambda \phi^{(i_1,0,0)}
+ \sum_{i_1=1}^{S} \sum_{i_3=1}^{N-1} r(N - i_3) \lambda \left( \phi^{(i_1,1,i_3)} + \phi^{(i_1,3,i_3)} \right)
+ \sum_{i_1=0}^{S} \sum_{i_3=1}^{N-1} r(N - i_3) \lambda \left( \phi^{(i_1,2,i_3)} + \phi^{(i_1,4,i_3)} \right).
$$

4.6 Expected loss rate for customers

Let $\eta_L$ denote the expected loss rate for the customers in the steady state, then

$$
\eta_L = \sum_{i_3=0}^{N-1} (1 - r)(N - i_3) \lambda \phi^{(0,0,i_3)} + \sum_{i_1=1}^{S} \sum_{i_3=1}^{N-1} (1 - r)(N - i_3) \lambda \left( \phi^{(i_1,1,i_3)} + \phi^{(i_1,3,i_3)} \right)
+ \sum_{i_1=0}^{S} \sum_{i_3=1}^{N-1} (1 - r)(N - i_3) \lambda \left( \phi^{(i_1,2,i_3)} + \phi^{(i_1,4,i_3)} \right).
$$
4.7 Effective interruption rate

Let $\eta_{\text{INTR}}$ denote the effective interruption rate which is given by

$$\eta_{\text{INTR}} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \alpha_1 \phi^{(i_1,1,i_3)} + \sum_{i_1=0}^{S} \sum_{i_3=1}^{N} \alpha_2 \phi^{(i_1,2,i_3)}.$$

4.8 Effective repair rate

Let $\eta_{\text{RR}}$ denote the effective repair rate which is given by

$$\eta_{\text{RR}} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \eta_1 \phi^{(i_1,3,i_3)} + \sum_{i_1=0}^{S} \sum_{i_3=1}^{N} \eta_2 \phi^{(i_1,4,i_3)}.$$

4.9 Probability that server is idle

Let $\eta_{\text{PI}}$ denote the probability that server is idle is given by

$$\eta_{\text{PI}} = \sum_{i_3=0}^{N} \phi^{(0,0,i_3)} + \sum_{i_1=1}^{S} \phi^{(i_1,0,0)}.$$

4.10 Probability that server is working

Let $\eta_{\text{PW}}$ denote the probability that server is working is given by

$$\eta_{\text{PW}} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} (\phi^{(i_1,1,i_3)} + \phi^{(i_1,3,i_3)}) + \sum_{i_1=0}^{S} \sum_{i_3=1}^{N} (\phi^{(i_1,2,i_3)} + \phi^{(i_1,4,i_3)}).$$

4.11 Probability that server is on FES

Let $\eta_{\text{PFES}}$ denote the probability that server is providing FES is given by

$$\eta_{\text{PFES}} = \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \phi^{(i_1,1,i_3)}.$$

4.12 Probability that server is on SOS

Let $\eta_{\text{SOS}}$ denote the probability that server is providing SOS is given by

$$\eta_{\text{PSOS}} = \sum_{i_1=0}^{S} \sum_{i_3=1}^{N} \phi^{(i_1,2,i_3)}.$$
5 Cost analysis

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

\[ TC(S, s, N) = c_h \eta_I + c_s \eta_R + c_p \eta_P + c_w \eta_W + c_l \eta_L + c_i \eta_{INT} + c_r \eta_{RR} \]

where

- \( c_h \): The inventory carrying cost per unit item per unit time
- \( c_s \): Setup cost per order
- \( c_p \): Perishable cost per unit item per unit time
- \( c_w \): Waiting cost of a customer per unit time
- \( c_l \): Cost per customer lost
- \( c_i \): Cost per interruption per unit time
- \( c_r \): Cost per repair per unit time

Substituting the values of \( \eta \)'s, we get \( TC(S, s, N) = \)

\[
\begin{align*}
&c_s \sum_{i_3=1}^{N} \mu_1 \phi^{(s+1, 1, i_3)} + c_s (s + 1) \gamma \phi^{(s+1, 0, 0)} + c_s \sum_{i_3=1}^{N} (s + 1) \gamma (\phi^{(s+1, 2, i_3)} + \phi^{(s+1, 4, i_3)}) \\
&+ c_s \sum_{i_3=1}^{N} s \gamma (\phi^{(s+1, 1, i_3)} + \phi^{(s+1, 3, i_3)}) + c_h \sum_{i_1=1}^{S} i_1 \phi^{(i_1)} e + c_p \sum_{i_1=1}^{S} i_1 \gamma \phi^{(s+1, 0, 0)}
\end{align*}
\]
\[
+c_p \sum_{i_1=2}^N \sum_{i_3=1}^N (i_1 - 1) \gamma (\phi^{(i_1,1,i_3)} + \phi^{(i_1,3,i_3)}) + c_p \sum_{i_3=1}^N i_1 \gamma (\phi^{(i_1,2,i_3)} + \phi^{(i_1,4,i_3)}) \\
+c_w \frac{\Gamma_1}{\Gamma_2} + c_l \sum_{i_3=0}^{N-1} (1 - r)(N - i_3) \lambda \phi^{(0,0,i_3)} + c_l \sum_{i_1=1}^N \sum_{i_3=1}^N (1 - r)(N - i - 3) \lambda (\phi^{(i_1,1,i_3)} + \phi^{(i_1,3,i_3)}) \\
+c_i \sum_{i_1=1}^S \sum_{i_3=1}^N \alpha_1 \phi^{(i_1,1,i_3)} + c_i \sum_{i_1=0}^N \sum_{i_3=1}^N \alpha_2 \phi^{(i_1,2,i_3)} + c_c \sum_{i_1=1}^N \sum_{i_3=1}^N \eta_1 \phi^{(i_1,3,i_3)} \\
+c_r \sum_{i_1=0}^S \sum_{i_3=1}^N \eta_2 \phi^{(i_1,4,i_3)}
\]

6 Numerical illustrations

In this section, some numerical examples that reveal the possible convexity of the total expected cost rate are discussed. A typical 3-dimensional plot of \( TC(S,s) \) is presented in Figure 1. The numerical search procedure is employed to obtain the optimal values of \( S, s \) and \( TC \) (say \( S^*, s^* \) and \( TC^* \)). The effect of varying the cost and other system parameters on the optimal values and the results agreed with what one would expect, have been studied. Some of the results are presented in Tables 4 through 11 where the lower entry in each cell gives the optimal expected cost rate and the upper entries the corresponding \( S^* \) and \( s^* \).

Example 1 First, the behaviour of the cost function is explored by considering as the function of two variables by fixing the others at a constant level. Tables 1 – 3, give the total expected cost rate as a function of \( TC(S,s,10), TC(50,s,N) \) and \( TC(S,7,N) \). All the costs and other parameters are assigned fixed values which are indicated in each Table. The value that is shown **bold** is the least among the values in that row and the value that is shown underlined is the least in that column. It may be observed that, these values in each Table exhibit a (possibly) local minimum of the function of the two variables. Also it may be observed that, the total expected cost rate function \( TC(S,s,N) \) is more sensitive to changes in \( N \) than to changes in \( S \) and \( s \).

Example 2 In this example, the impact of the setup cost \( c_s \), holding cost \( c_h \), waiting cost \( c_w \), shortage cost \( c_l \), perishable cost \( c_p \), interruption cost \( c_i \) and repair cost \( c_r \) on the optimal values (possibly local) \( S^*, s^* \) and \( TC^* \) is studied. Towards this end, the parameter values as \( \lambda = 0.7, \beta = 0.1, \gamma = 1, \alpha_1 = 0.3, \alpha_2 = 0.14, \mu_1 = 0.8, \mu_2 = 0.9, \eta_1 = 9, \eta_2 = 7, p = 0.7, q = 0.5, r = 0.4, N = 10 \) are first fixed. The following from Table 4 to 11 are observed:
the following.

Example 3 In this example, we look to the impact of the demand rate $\lambda$, essential service rate $\mu_1$, second optional service rate $\mu_2$, the reorder rate $\beta$ and the perishable rate $\gamma$ on the total expected cost rate. Towards this end, we first fix the cost values as $c_h = 0.01$, $c_s = 50$, $c_p = 0.1$, $c_w = 0.03$, $c_t = 0.2$, $c_i = 0.1$, $C_r = 0.4$. From Figures 2 to 6, we observe the following.

### Table 1: Total expected cost rate as a function of $S$ and $s$

<table>
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<tr>
<th>$S$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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</tr>
<tr>
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</tr>
<tr>
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<table>
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<td>3.861753</td>
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</table>

1. The total expected cost rate monotonically increases when $c_h$, $c_s$, $c_w$, $c_p$, $c_t$, $c_i$ and $c_r$ increase. The optimal cost is more sensitive to $c_h$ than to $c_s$, $c_w$, $c_p$, $c_t$, $c_i$ and $c_r$.
2. As is to be expected, as $c_h$ increases, the optimal values $S^*$ and $s^*$ decrease monotonically. This is because, if the holding cost increases, we resort to maintain low stock in the inventory.
3. If the setup cost increases, it is a common decision that we have to maintain more stock to avoid frequent ordering. This fact is also observed in the model.
4. If the waiting cost $c_w$ of customers increases then the optimal values $S^*$ and $s^*$ monotonically increase. This is because if waiting cost of customers increases then we have to maintain high inventory to reduce the number of waiting customers. Also, we note that $c_p$, $c_t$, $c_i$ and $c_r$ monotonically increase when $S^*$ decreases.
5. As is to be expected as $s^*$ decreases, $c_p$ increases. We cannot predict the behaviour of $s^*$ when each of $c_t$, $c_i$ and $c_r$ increases.
Finite source perishable inventory with second optional service and server interruptions

Table 3: Total expected cost rate as a function of $S$ and $N$

\[ \lambda = 0.07, \beta = 0.1, \gamma = 1, \alpha_1 = 3, \alpha_2 = 0.14, \mu_1 = 0.8, \mu_2 = 9, \eta_1 = 9, \eta_2 = 7, p = 0.7, q = 0.3, r = 0.5, c_h = 0.01, c_s = 50, c_p = 0.1, c_w = 0.03, c_l = 0.2, c_i = 0.1, C_r = 0.4. \]

<table>
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<tr>
<th>$N$</th>
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Table 4: Variation in optimal values for different values of $c_h$ and $c_s$

\[ c_p = 0.1, c_w = 0.03, c_l = 0.2, c_i = 0.1, C_r = 0.4. \]

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</table>

- The optimal expected cost rate increases when $\lambda$ and $\gamma$ increase.
- As is to be expected, the optimal cost rate decreases, when $\beta, \mu_1$ and $\mu_2$ decrease.

Example 4 In this example, we look to the impact of the demand rate $\lambda$, essential service rate $\mu_1$, second optional service rate $\mu_2$, waiting hall size $N$, the interruption rates $\alpha_1$ and $\alpha_2$ during FES and SOS respectively, repair rates $\eta_1$ and $\eta_2$ during FES and SOS respectively, on the expected waiting time. Towards this end, we first fix the cost values as $c_h = 0.01, c_s = 50, c_p = 0.1, c_w = 0.03, c_l = 0.2, c_i = 0.1, C_r = 0.4$. From Figures 7 to 11, we observe the following.

- The expected waiting time $\eta_W$ is an increasing function of arrival rate (see Figure 6) and this behaviour is maintained for various values of $N = 10, 20, 30$. However, the expected waiting time is higher if $N$ is larger.
- The expected waiting time increases when $\alpha_1$ and $\alpha_2$ decrease.
- As is to be expected, waiting time cost $\eta_W$, decreases, when $\eta_1, \eta_2, \mu_1$ and $\mu_2$ increase.
<table>
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Table 5: Variation in optimal values for different values of $c_h$ and $c_p$

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Table 7: Variation in optimal values for different values of $c_p$ and $c_s$

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Table 8: Variation in optimal values for different values of $c_w$ and $c_s$

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Table 10: Variation in optimal values for different values of $c_p$ and $c_l$

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\( N = 10, \ p = 0.3, \ q = 0.7, \ r = 0.5 \)

**Figure 2:** \( TC \) versus \( \beta \) for different values of \( \lambda \)

\( \gamma = 0.001, \gamma = 0.004, \gamma = 0.007; \)
\( N = 10, \ \lambda = 7, \ \alpha_1 = 0.003, \ \alpha_2 = 0.1, \ \mu_1 = 0.8, \ \mu_2 = 9, \ \eta_1 = 9, \ \eta_2 = 7, \ p = 0.3, \ q = 0.7, \ r = 0.5 \)

**Figure 3:** \( TC \) versus \( \beta \) for different values of \( \gamma \)
Finite source perishable inventory with second optional service and server interruptions

\[ \lambda = 0.7, \; \gamma = 0.1, \; \alpha_1 = 0.3, \; \alpha_2 = 0.1, \; \mu_2 = 9, \; \eta_1 = 9, \; \eta_2 = 0.7, \]
\[ p = 0.3, \; q = 0.7, \; r = 0.5, \; N = 10 \]

**Figure 4:** TC versus $\beta$ for different values of $\mu_1$

\[ \lambda = 5, \; \beta = 0.01, \; \alpha_1 = 0.3, \; \alpha_2 = 0.1, \; \mu_2 = 3, \; \eta_1 = 9, \; \eta_2 = 7, \]
\[ p = 0.3, \; q = 0.7, \; r = 0.5, \; N = 10 \]

**Figure 5:** TC versus $\gamma$ for different values of $\mu_1$
Table 11: Variation in optimal values for different values of $c_r$ and $c_i$

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$\lambda = 5$, $\beta = 0.01$, $\alpha_1 = 0.3$, $\alpha_2 = 0.1$, $\mu_1 = 5$, $\mu_2 = 9$, $\eta_1 = 9$, $\eta_2 = 7$, $p = 0.3$, $q = 0.7$, $r = 0.5$, $N = 10$

Figure 6: $TC$ versus $\gamma$ for different values of $\mu_2$
\[ \beta = 0.1, \; \gamma = 1, \; \alpha_1 = 0.3, \; \alpha_2 = 0.14, \; \mu_1 = 0.8, \; \mu_2 = 9, \; \eta_1 = 9, \; \eta_2 = 0.7, \; p = 0.7, \; q = 0.3, \; r = 0.5, \; N = 10 \]

**Figure 7:** \( \eta_W \) versus \( \lambda \) for different values of \( N \)

\[ \lambda = 0.7, \; \beta = 0.1, \; \gamma = 1, \; \mu_1 = 0.8, \; \mu_2 = 9, \; \eta_1 = 9, \; \eta_2 = 0.7, \; p = 0.7, \; q = 0.3, \; r = 0.5, \; N = 10 \]

**Figure 8:** \( \eta_W \) versus \( \alpha_1 \) for different values of \( \alpha_2 \)
$\lambda = 0.7$, $\beta = 0.1$, $\gamma = 1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.14$, $\mu_1 = 0.8$, $\mu_2 = 9$, $p = 0.7$, $q = 0.3$, $r = 0.5$, $N = 10$

**Figure 9:** $\eta_W$ versus $\eta_1$ for different values of $\eta_2$

$\lambda = 0.7$, $\beta = 0.1$, $\gamma = 1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.14$, $\mu_2 = 9$, $\eta_1 = 9$, $\eta_2 = 0.7$, $p = 0.7$, $q = 0.3$, $r = 0.5$

**Figure 10:** $\eta_W$ versus $\mu_1$ for different values of $N$
$\lambda = 0.7, \beta = 0.1, \gamma = 1, \alpha_1 = 0.3, \alpha_2 = 0.14, \mu_1 = 0.8, \eta_1 = 9, \eta_2 = 0.7,$
\[ p = 0.7, \quad q = 0.3, \quad r = 0.5 \]

**Figure 11:** $\eta_W$ versus $\mu_2$ for different values of $N$

### 7 Summary and conclusion

In this article, a continuous review stochastic queueing-inventory system with $(s, S)$ control policy, server interruptions and finite source was analyzed. The model is analyzed within the framework of Markov processes. Stationary distribution of the number of customers in the waiting area, the server status and the inventory level is obtained in the steady state. Various system performance measures are derived and the long-run total expected cost rate is derived. The waiting time distribution is derived. A sensitivity analysis is numerically performed on the expected total cost function with respect to various parameters of the model. The authors are working in the direction of MAP (Markovian arrival process) arrival for the customers and service times follow PH-distributions.

### Acknowledgement

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### References


